PRIMES Math Problem Set: Solutions

PRIMES

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Solution to General Math Problems

Problem G1

We have n fair six-sided dice, labeled 1 through 6. Let p_n be the probability that when rolled, the product of all n numbers shown is at most 6.

(a) Compute the value of p_2.

(b) Determine p_n for any integer $n \geq 2$.

Solution

We proceed by enumerating all possible multisets of n dice rolls with product at most six.

- $1^n = 1$, which occurs with probability 6^{-n}.
- For $2 \leq k \leq 6$, we have $1^{n-1} \cdot k = k$, which occurs with probability $\binom{n}{1} \cdot 6^{-n}$.
- $1^{n-2} \cdot 2^2 = 4$ occurs with probability $\binom{n}{2} \cdot 6^{-n}$.
- $1^{n-2} \cdot 2 \cdot 3 = 6$ occurs with probability $n(n-1) \cdot 6^{-n}$.

Putting these together gives

$$\begin{aligned} p_n &= 6^{-n} + 5 \cdot n \cdot 6^{-n} + \binom{n}{2} \cdot 6^{-n} + n(n-1) \cdot 6^{-n} \\ &= 6^{-n} \left(1 + 5n + 3\binom{n}{2}\right) \\ &= \frac{3n^2 + 7n + 2}{2 \cdot 6^n}. \end{aligned}$$

In particular, $p_2 = \frac{7}{18}$.

Problem G2

Fix an integer $d \geq 1$, and consider polynomials $P(x)$ of degree d which satisfy

$$P(n) = n + \frac{1}{n}$$

for $n = 1, 2, \ldots, 99$. For each d, determine all possible values of $P(100)$ or show that no such polynomials P of degree d exist.

Solution

Such a polynomial exists for $d \geq 98$. Moreover,

- When $d = 98$, the unique solution is $P(100) = 100.5$.
- When $d = 99$, $P(100)$ can take any value other than 100.5.
- When $d \geq 100$, $P(100)$ can take any value.

The condition is equivalent to

$$Q(n) = nP(n) - (n^2 + 1)$$

having roots at $n = 1, 2, \ldots, 99$. Thus, this implies that $\deg Q \geq 99$, so $d = \deg P \geq 98$.

If $d = 98$, or $\deg Q = 98$, then we have that

$$nP(n) - (n^2 + 1) = c(n-1)(n-2)\ldots(n-99)$$

for some constant c. Choosing $n = 0$, we have $99! \cdot (-c) = -1$, so $c = \frac{1}{99!}$. Hence, in the $d = 98$ we must have

$$P(n) = \frac{1}{n}\left[(n^2 + 1) + \frac{(n-1)(n-2)\ldots(n-99)}{99!}\right].$$

At $P(100)$ we get $P(100) = \frac{1}{100}\left(100^2 + 1 + 1\right) = 100.5$.

When $d \geq 100$ on the other hand, any polynomial of the form

$$P(n) = \frac{1}{n}\left[(n^2 + 1) + \frac{(n-1)(n-2)\ldots(n-99)}{99!}\right] + (n-1)(n-2)\ldots(n-99)R(n)$$

will work for any polynomial $R(n)$ of suitable degree. So $P(100)$ can take any value for $d \geq 100$.

When $d = 99$, by taking $R(n)$ to be a suitable nonzero constant as above, we see that $P(100)$ takes any value other than 100.5. On the other hand, there is no P of degree 99 which $P(100) = 100.5$ since by e.g. Lagrange interpolation there is at most one such polynomial of degree at most 99, but it has degree 98.

Problem G3

There are 100 marbles in a bag: 30 red marbles, 60 green marbles and 10 yellow marbles. We select three of them uniformly at random, independently and with replacement (meaning we put the ball back after it is removed). Let E be the event "all three marbles are the same color".

(a) Find the probability of E.

(b) Find the probability of E, given that at least one selected marble is red.

(c) Find the probability of E, given that the selected marbles aren't all different colors.

Solution

(a) This is $0.3^3 + 0.6^3 + 0.1^3 = 0.254$.

(b) The probability that none of them are red is $0.7^3 = 0.343$ so the probability that at least one is red $1 - 0.7^3 = 0.657$. On the other hand the probability all three are red is 0.027, so the answer is $27/657 = 3/73$.

(c) The probability that no two are the same color is given by $3! \cdot 0.3 \cdot 0.6 \cdot 0.1 = 0.108$, so the probability that some two are the same color is $1 - 3! \cdot 0.3 \cdot 0.6 \cdot 0.1 = 0.892$. The probability all three are the same color was given in (a) as 0.254. Hence the answer is $254/892 = 127/446$.

Problem G4

A pair (σ, τ) of permutations on $\{0, 1, \ldots, n-1\}$ is *balanced* if the following map is also a bijection on $\{0, 1, \ldots, n-1\}$:

$$x \mapsto (\sigma(x) + \tau(x)) \mod n.$$

(Here, $a \mod n$ means the remainder when a is divided by n.)

(a) Does there exist a balanced pair when $n = 3$? (Either give an example or prove none exists.)

(b) Does there exist a balanced pair when $n = 4$? (Either give an example or prove none exists.)

(c) For which n does there exist a balanced pair?

Solution

We'll prove that a balanced pair exists if and only if n is odd.

Assume for contradiction a balanced pair exists for even n. Summing the function $\sigma + \tau$ across $x = 0, 1, \ldots, n-1$ gives

$$0 + 1 + \cdots + (n-1) \equiv 2(0 + 1 + \cdots + (n-1)) \pmod n$$

and we conclude n divides $0 + 1 + \cdots + (n-1) = \frac{1}{2}n(n-1)$; in other words we expect

$$\frac{\frac{1}{2}n(n-1)}{n} = \frac{n-1}{2}$$

to be an integer. But if n is even, then $n-1$ is odd so this is wrong.

As for odd n, one can take $\sigma(x) = \tau(x) = x$. Then the sum of all of these is $x \mapsto 2x \mod n$, which is a bijection for n odd.

Problem G5

Consider the following six points in the coordinate plane:

$$A = (0, 1), \quad B = (0, 3), \quad C = (1, 4), \quad D = (4, 9), \quad E = (6, 7), \quad F = (6, 8).$$

For a point P in the coordinate plane let $S(P) = PA + PB + PC + PD + PE + PF$.

(a) Prove that $S(P)$ is minimized at some point P.

(b) Determine the value of that minimum.

Solution

For any point P we have the following inequalities:

$$\begin{aligned} PA + PD &\geq AD \\ PB + PF &\geq BF \\ PC + PE &\geq CE \end{aligned}$$

Summing all of these gives a lower bound

$$S(P) \geq AD + BF + CE = 4\sqrt{5} + \sqrt{61} + 2\sqrt{13}.$$

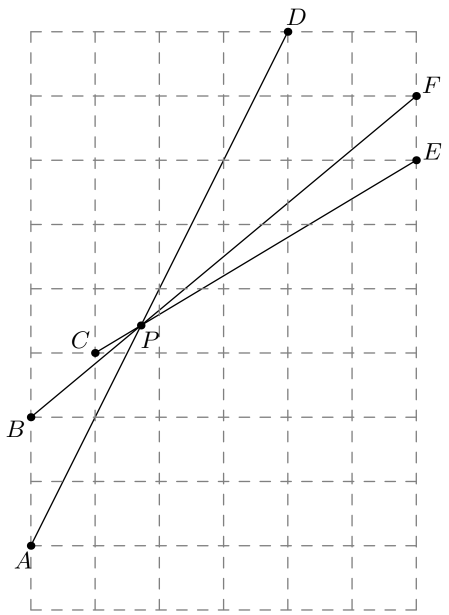
This is achieved if P lies on all the segments AD, BF, CE.

We claim this occurs for the point

$$P = \left(\frac{12}{7}, \frac{31}{7}\right)$$

(which must be unique). Indeed, we have

$$P = \frac{4}{7}A + \frac{3}{7}D = \frac{5}{7}B + \frac{2}{7}F = \frac{6}{7}C + \frac{1}{7}E.$$



One can prove (a) without solving (b) in the following manner: we restrict our attention to a closed disk $\mathcal{D}$ centered at the origin $O = (0, 0)$ with radius $R = S(O) + 1000$, since points outside this disk will clearly have $S(P) > S(O)$. Then S is a continuous function on $\mathcal{D}$ which is bounded below (by zero), so by compactness, there exists a minimum value of S over $\mathcal{D}$.

Problem G6

Let a, b, c be positive real numbers for which $\min(ab, bc, ca) \geq 1$.

(a) Prove that $\log(abc) \geq \sqrt[3]{(\log a)^3 + (\log b)^3 + (\log c)^3}$.

(b) Determine for which triples (a, b, c) the equality holds.

Solution

Let $x = \log a$, $y = \log b$, $z = \log c$. Now

$$x + y = \log(a) + \log(b) = \log(ab) \geq 0.$$

Similarly $y + z \geq 0$, $z + x \geq 0$. Then we are done with (a) upon writing

$$(x + y + z)^3 - (x^3 + y^3 + z^3) = 3(x + y)(y + z)(z + x) \geq 0.$$

The equality is sharp whenever any of $\{x + y, y + z, z + x\}$ vanishes, i.e. whenever $\min(ab, bc, ca) = 1$.

Problem G7

For each integer $n \geq 1$ let $\mathcal{T}_n$ denote the set of nondegenerate triangles whose side lengths are in $\{1, \ldots, n\}$. Moreover, for each triangle $\triangle ABC$, let

$$D(\triangle ABC) = \min\left(|AB - AC|, |BC - BA|, |CA - CB|\right).$$

(a) For each integer $n \geq 3$, determine the largest possible value of $D(\triangle ABC)$ over all triangles in $\mathcal{T}_n$.

(b) For which n is this maximum value achieved for a unique triangle in $\mathcal{T}_n$ (up to congruence)?

Solution

We claim that $D(\triangle ABC) = \left\lfloor \frac{n-1}{3} \right\rfloor$ is optimal, and that for part (b) the answer is $n \equiv 1 \pmod 3$.

Throughout, let $\delta = D(\triangle ABC)$ for brevity, and let $a \geq b \geq c$ denote the side lengths.

First, we show that one cannot take δ any larger. By the triangle inequality, we have $b + c > a$, or $b + c \geq a + 1$. Then we write

$$\begin{aligned}
(n - \delta) + c &\geq (a - \delta) + c \\
&\geq b + c \geq a + 1 \\
&= (a - b) + (b - c) + c + 1 \\
&\geq c + 2\delta + 1.
\end{aligned}$$

This shows that $3\delta \leq n - 1$, as desired.

We give two examples each when $n \equiv 0, 2 \pmod 3$.

- If $n = 3k$, then the triangles $(3k, 2k + 1, k + 2)$ and $(3k, 2k + 1, k + 1)$ both have $\delta = k - 1$.
- If $n = 3k + 2$, then the triangles $(3k + 2, 2k + 2, k + 2)$ and $(3k + 2, 2k + 2, k + 1)$ both have $\delta = k$.

Finally, for $n = 3k + 1$ we have $(3k + 1, 2k + 1, k + 1)$. To see this is tight, note that in the displayed equations, we must have equality everywhere, meaning $a = n$, and $a - b = b - c = \delta$ (and also $a + 1 = b + c$). Since $\delta = k$ and $a = 3k + 1$, that implies the result.

Solution to Advanced Math Problems

Problem M1

Let P be a partially ordered set (poset) with 12 elements. Given that P has width 2, what is the maximum number of linear extensions that P can have?

(A *linear extension* of a poset P is a total ordering of the elements compatible with the partial order. The *width* of a partially ordered set is the largest size of a subset in which no two elements are comparable; this is the size of the largest antichain. See `https://en.wikipedia.org/wiki/Partially_ordered_set` for the definition of a poset.)

Solution

By Dilworth's theorem, since P has width 2 it may be partitioned into two chains, say

$$a_1 \leq \cdots \leq a_k \qquad \text{and} \qquad b_1 \leq \cdots \leq b_{12-k}.$$

If there are any relations between a_i and b_j, then we can delete them without decreasing the width of the poset, but increasing the number of linear extensions. Hence we can reduce to the case where P is the disjoint union of two chains as above.

In that case, a linear extension of P corresponds to a way choose k positions out of 12 for an a_i, and assign the remaining $12 - k$ to b_i's. The number of such situations is $\binom{12}{k}$. This is maximized at $\binom{12}{6} = 924$.

Problem M2

Consider the infinite series

$$S = \sum_{n=2}^{\infty} \left[\log(n^3 + k) - \log(n^3 - k)\right].$$

where $k \in (0, 8)$ is a real number. (Here log denotes the natural logarithm.)

(a) Prove that S converges for any k.

(b) For $k = 1$, compute S.

Solution

For part (a), write

$$S = \sum_{n \geq 2} \log\left(\frac{n^3 + k}{n^3 - k}\right).$$

This is a series of positive real numbers, so it suffices to show the partial sums are bounded above. Then in light of the estimate $\log(x) \leq x - 1$, it is enough to show that

$$\sum_{n \geq 2} \left(\frac{n^3 + k}{n^3 - k} - 1\right) = \sum_{n \geq 2} \left(\frac{2k}{n^3 - k}\right) < \infty.$$

Now for all $n \geq 3$ we have $n^3 - k > n^2$ so we may bound the above sum by

$$2k\left(\frac{1}{8-k} + \sum_{n \geq 3} \frac{1}{n^2}\right) < \infty$$

since $\sum_{n \geq 1} n^{-2} = \frac{\pi^2}{6}$. This concludes the proof of (a).

As for (b), the partial sums telescope:

$$\begin{aligned}
S &= \lim_{N \to \infty} \sum_{n=2}^{N} \big(\log(n+1) + \log(n^2 - n + 1) - \log(n-1) - \log(n^2 + n + 1)\big) \\
&= \lim_{N \to \infty} \big([\log(N+1) + \log(N) - \log(2) - \log(1)] - \big[\log(N^2 + N + 1) - \log(3)\big]\big) \\
&= \lim_{N \to \infty} \left(\log(3/2) + \log\left(\frac{N^2 + N}{N^2 + N + 1}\right)\right) \\
&= \log(3/2) + \log(1).
\end{aligned}$$

$$\implies S = \log(3/2).$$

Problem M3

Consider integrable functions $f : [0, \pi] \to [-1, 1]$ such that $\int_0^\pi f(x) = 0$, and let

$$S(f) = \int_0^\pi f(x) \sin x \, dx.$$

Find a constant M, as small as you can, for which $|S(f)| \le M$.

Solution

By replacing f with $-f$ when appropriate, it's enough to consider $S(f) \le M$.

Let g be the function defined by

$$g(x) = \begin{cases} 1 & \frac{1}{4}\pi \le \frac{3}{4}\pi \\ -1 & \text{otherwise.} \end{cases}$$

We have that

$$\begin{aligned} S(g) = \int_0^\pi g(x) \sin x \, dx &= -\int_0^{\frac{\pi}{4}} \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x \, dx - \int_{\frac{3\pi}{4}}^{\pi} \sin x \, dx \\ &= -\left(1 - \frac{1}{\sqrt{2}}\right) + \sqrt{2} - \left(1 - \frac{1}{\sqrt{2}}\right) \\ &= 2\sqrt{2} - 2. \end{aligned}$$

Hence $M \ge 2\sqrt{2} - 2$.

We claim that this is best possible. To see this, note that

$$\begin{aligned} S(g) - S(f) &= \int_0^\pi [g(x) - f(x)] \sin x \, dx \\ &= \int_0^{\frac{\pi}{4}} \underbrace{[g(x) - f(x)]}_{\le 0} \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \underbrace{[g(x) - f(x)]}_{\ge 0} \sin x \, dx \\ &\quad + \int_{\frac{3\pi}{4}}^{\pi} \underbrace{[g(x) - f(x)]}_{\le 0} \sin x \, dx \\ &\ge \int_0^{\frac{\pi}{4}} \underbrace{[g(x) - f(x)]}_{\le 0} \sin(\pi/4) \, dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \underbrace{[g(x) - f(x)]}_{\ge 0} \sin(\pi/4) \, dx \\ &\quad + \int_{\frac{3\pi}{4}}^{\pi} \underbrace{[g(x) - f(x)]}_{\le 0} \sin(\pi/4) \, dx \\ &= \sin(\pi/4) \int_0^\pi [g(x) - f(x)] \, dx \\ &= 0. \end{aligned}$$

Problem M4

For integers $n \geq 0$ let
$$a_n = \sum_{i=0}^{n} \frac{1}{i+1} \binom{n+i}{n-i} \binom{2i}{i}.$$
Identify the sequence $(a_n)_n$ by name and prove that a_n is the claimed sequence. (You may use the Online Encyclopedia of Integer Sequences, `http://oeis.org/`.)

Solution

The answer is that a_n is the nth Schroeder number. Our proof is by generating functions. We write

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} \left(\sum_{i=0}^{n} \frac{1}{i+1} \binom{n+i}{2i} \binom{2i}{i} x^n \right) \\
&= \sum_{i\geq 0} \left(\frac{1}{i+1} \binom{2i}{i} \sum_{n\geq i} \binom{n+i}{2i} x^n \right) \\
&= \sum_{i\geq 0} \left(\frac{1}{i+1} \binom{2i}{i} x^i \sum_{k\geq 0} \binom{k+2i}{2i} x^k \right) \\
&= \sum_{i\geq 0} \left(\frac{1}{i+1} \binom{2i}{i} x^i \frac{1}{(1-x)^{2i+1}} \right) \\
&= \frac{1}{1-x} \sum_{i\geq 0} \left(\frac{1}{i+1} \binom{2i}{i} \left(\frac{x}{(1-x)^2} \right)^i \right) \\
&= \frac{1}{1-x} \cdot \frac{2}{1+\sqrt{1-4\frac{x}{(1-x)^2}}} \\
&= \frac{2}{1-x+\sqrt{(1-x)^2-4x}} \\
&= \frac{2}{1-x+\sqrt{x^2-6x+1}}
\end{aligned}$$

which is the generating function of the Schroeder numbers. Here we have used the Catalan number generating function $\sum_i \frac{1}{i+1}\binom{2i}{i} X^i = \frac{2}{1+\sqrt{1-4X}}$.

Problem M5

Let G be a group with presentation given by

$$G = \left\langle a, b, c \mid ab = c^2a^4,\ bc = ca^6,\ ac = ca^8,\ c^{2018} = b^{2019} \right\rangle.$$

(a) Show that G is finite.

(b) Determine the order of G.

Solution

First, observe that by induction we have

$$a^n c = ca^{8n}$$

for all $n \geq 1$. We then note that

$$\begin{aligned} a(bc) &= (ab)c \\ a \cdot ca^6 &= c^2a^4 \cdot c \\ ca^8 \cdot a^6 &= c^2a^4 \cdot c \\ a^{14} &= c(a^4c) = c^2a^{32}. \end{aligned}$$

Hence we conclude $c^2 = a^{-18}$. Then $ab = c^2a^4 \implies b = a^{-15}$.

In that case, if $c^{2018} = b^{2019}$, we conclude $1 = a^{2018\cdot 18 - 2019\cdot 15} = a^{6039}$. Finally,

$$\begin{aligned} bc &= ca^6 \\ a^{-15}c &= ca^6 \\ a^{-15}c^2 &= c(a^6c) = c^2a^{48} \\ a^{-33} &= a^{30} \\ \implies a^{63} &= 1. \end{aligned}$$

Since $\gcd(6039, 63) = 9$, we find $a^9 = 1$, hence finally $c^2 = 1$. So the presentation above simplifies to

$$G = \left\langle a, c \mid a^9 = c^2 = 1,\ ac = ca^{-1} \right\rangle$$

which is the presentation of the dihedral group of order 18.

Problem M6

Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix. Assume that $a_{ij} \le 0$ for $i \neq j$. Show that the following two conditions are equivalent:

- The matrix A is positive-definite.
- There exists a vector v such that both v and Av have strictly positive entries.

Solution

In what follows, by $v > 0$ we mean the entries of v are positive.

First, we prove that if A is positive definite then $v > 0$ with the desired property exists. Actually, we will prove the following claim, which implies the result since we can then take $v = A^{-1}w$ for any $w > 0$.

Claim: If $A = (a_{ij})$ is a symmetric positive definite $n \times n$ matrix with nonpositive entries off the diagonal, then all entries of A^{-1} are nonnegative, and the diagonal entries are positive.

Proof. We proceed by induction on n, with the base case being clear. Let A_{ij} denote the matrix obtained when the ith row and jth column of A are deleted. Since $\det A > 0$, it's equivalent to check all the cofactors of A are nonnegative, as

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn}. \end{bmatrix}$$

where $C_{ij} = (-1)^{i+j} \det A_{ij}$ is the (i,j)-cofactor.

Since A is positive definite we immediately have $C_{ii} = \det A_{ii} > 0$.

Now we show $C_{12} \ge 0 \iff \det A_{12} \le 0$, say. Let B_{ij} be the matrix obtained by deleting from A_{12} the ith row and jth column of A. Then

$$\det A_{12} = \sum_{i=2}^{n} (-1)^i a_{i1} \det B_{i1}.$$

But B_{i1} is the $(i-1,1)$-minor of the matrix A_{11}. As the matrix A_{11} is also positive definite (it's a principal minor and induction hypothesis applies), we have $(-1)^i \det B_{i1} \ge 0$. Then $a_{i1} \le 0$, hence $(-1)^i a_{i1} \det B_{i1} \le 0$ for each i, done. $\square$

Now conversely assume $v = (v_1, \dots, v_n) > 0$ is given so that $Av > 0$; we prove every eigenvalue λ of A is positive. This is essentially a modified proof of the Gershgorin circle theorem, modified to fit our context.

Write the eigenvector as $(v_1 y_1, \dots, v_n y_n)$. For any k,

$$\sum_j a_{kj} v_j y_j = \lambda v_k y_k \implies v_k (a_{kk} - \lambda) y_k = \sum_{j \neq k} -a_{kj} v_j y_j.$$

Now assume WLOG that $1 = y_k = \max_i y_i$. Then the triangle inequality implies that

$$v_k |a_{kk} - \lambda| \le \sum_{j \neq k} -a_{kj} v_j < a_{kk} v_k$$

where the last inequality follows from the hypothesis $\sum_j a_{kj} v_j > 0$ (from $Av > 0$). Hence $\lambda > 0$ as desired.