Behavior of Bar-Natan Homology under Conway Mutation

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Abstract

The Bar-Natan homology is a perturbation of the Khovanov homology of a knot. Previous work has shown that Khovanov homology remains unchanged under Conway mutation of the knot diagram. We give an exact triangle with three different resolutions of a link and prove several lemmas relating the dimensions of different Bar-Natan chain complexes and homologies. These allow us to prove that the dimension of the Bar-Natan homology $\mathrm{BN}^k(L; \mathbb{Z}/2\mathbb{Z})$ is invariant under Conway mutation.

1 Introduction

Knots are perhaps some of the least abstract mathematical objects, as they are commonly found in real life. Mathematically, however, a knot is simply an embedding of a circle in $\mathbb{R}^3$. Knot theory, the study of such knots, remains an active field of mathematical research to date. Knot theory has many applications in fields of science, including the chemistry of DNA and molecular knots. [1]

A knot diagram is a projection of a knot onto a plane. A knot diagram can contain crossings, or places where the knot crosses itself. Mathematicians consider two knots to be essentially the same knot if one can continuously deform one knot into the other. Hence the natural definition of a knot invariant: some quantity that is the same for identical knots (up to deformation). In 1984, mathematician Vaughan Jones discovered a knot invariant, the Jones polynomial [5], that brought forth a rush of new interest in knot theory, where many mathematicians attempted to generalize or apply the Jones polynomial. In the year 2000, mathematician Mikhail Khovanov succeeded in categorizing the Jones polynomial with the introduction of a new knot invariant, Khovanov homology [6]. The Khovanov homology has the property that its graded Euler characteristic is the Jones Polynomial. In 2002, mathematician Dror Bar-Natan defined a variant of Khovanov homology and proved that this Bar-Natan homology was also a knot invariant [3].

Conway mutations are one of many ways to transform knots. A Conway mutation takes a tangle of the knot and rotates it 180 degrees, then glues it back into the knot. We note that all mutant pairs can be drawn in the form shown in Figure 1, as proved in [7].

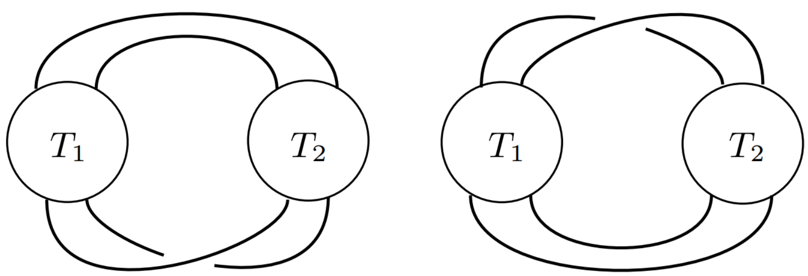


Figure 1: Mutant knots in standard form. Picture from [7].

Recent work from Bloom and Wehrli has shown that the Khovanov Homology is invariant under Conway mutation (up to isomorphism) [4]. Lambert-Cole even more recently outlined a simpler argument for the mutation invariance of Khovanov Homology in [7].

In this paper, we focus on the behavior of Bar-Natan homology over $F = \mathbb{Z}/2\mathbb{Z}$ under Conway mutation. Specifically, we construct an exact triangle relating various resolutions of a link in Lemma 4. By proving several relations on the dimensions of the Bar-Natan chain complexes and homologies, we are able to use the exact triangle in succession to prove our main result, Theorem 8: for any $k \geq 1$, $\dim(\mathrm{BN}^k(L))$ is invariant under Conway mutation.

2 Notation

For precise definitions of the Khovanov homology and the Bar-Natan homology, refer to [9] and [8]. The field we use is $F = \mathbb{Z}/2\mathbb{Z}$, so we may eliminate sign issues.

Throughout this paper we notate $\mathrm{BN}(L)$ as the unreduced Bar-Natan homology, and $\widetilde{\mathrm{BN}}(L)$ as the reduced homology for a link L. The reduced version of the Bar-Natan homology is defined as follows: we place a point (known as the basepoint) on any part of the knot; in a complete resolution of the knot, we only choose one of the two generators (specifically, v_+) from the circle which contains the basepoint.

Recall that we create the Khovanov homology by taking the homology of the Khovanov chain complex, which is constructed from a resolution cube, using specific merge and split formulas for the differential ([9]). We now alter the complex by tensoring it with $\mathbb{F}[U]$, and then change the merge and split formulas as shown in [8], to make the Bar-Natan chain complex, which we can take the homology of to make the Bar-Natan homology. We denote $C_{\mathrm{BN}}(L)$, $\widetilde{C}_{\mathrm{BN}}(L)$ as the unreduced and reduced Bar-Natan chain complexes, respectively. BN^k, $\widetilde{\mathrm{BN}}^k$, C^k_{BN}, and $\widetilde{C}^k_{\mathrm{BN}}$ are defined similarly, but the ring is $\mathbb{F}[U]/U^k$ instead of $\mathbb{F}[U]$ (so that the superscript k simply denotes a change in the ring). For each of the following results, unless otherwise stated, the result holds true for all $k \geq 1$.

Finally, in regards to mutation of links, $L_1 \cup L_2$ denotes the disjoint union of links L_1 and L_2, considered as one link; see Figure 2a. $L_1 \# L_2$ denotes the connected sum of L_1 and L_2 as in Figure 2b.

3 Extending the Argument

We follow Lambert-Cole's proof in [7] with a few modifications that render the argument suitable for Bar-Natan Homology. To do so, we prove the many preliminary results below. Throughout this section, L is a link.

To start, we note that the reduced version of the Bar-Natan homology has some nice properties, as in this result from [8]:

Lemma 1. $\widetilde{\mathrm{BN}}^k(L)$ *does not depend on the choice of basepoint.*

This fact allows us to relate the reduced and unreduced versions of the Bar-Natan homology:

Lemma 2. $\dim(\mathrm{BN}^k(L)) = 2\dim(\widetilde{\mathrm{BN}}^k(L))$.

Proof. Let N denote the unknot. We consider $\widetilde{C}^k_{\mathrm{BN}}(L \cup N)$. We can put the basepoint on N or L, leading to the following, respectively:

$$\widetilde{C}^k_{\mathrm{BN}}(L \cup N) \cong C^k_{\mathrm{BN}}(L),$$

and

$$\widetilde{C}^k_{\mathrm{BN}}(L \cup N) \cong \widetilde{C}^k_{\mathrm{BN}}(L) \oplus \widetilde{C}^k_{\mathrm{BN}}(L).$$

When taking homology, we obtain:

$$\mathrm{BN}^k(L) = \widetilde{\mathrm{BN}}^k(L) \oplus \widetilde{\mathrm{BN}}^k(L),$$

which implies the result. □

Lemma 3. *Given any two disjoint links L_1 and L_2, we have:*

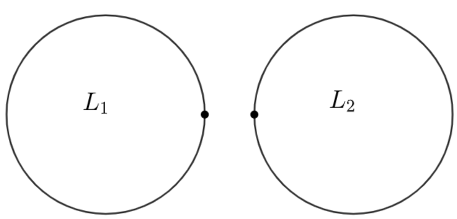
1. $$C^k_{\mathrm{BN}}(L_1 \cup L_2) \cong C^k_{\mathrm{BN}}(L_1) \underset{\mathbb{F}[U]/U^k}{\otimes} C^k_{\mathrm{BN}}(L_2).$$

2. $$\widetilde{C}^k_{\mathrm{BN}}(L_1 \# L_2) \cong \widetilde{C}^k_{\mathrm{BN}}(L_1) \underset{\mathbb{F}[U]/U^k}{\otimes} \widetilde{C}^k_{\mathrm{BN}}(L_2).$$

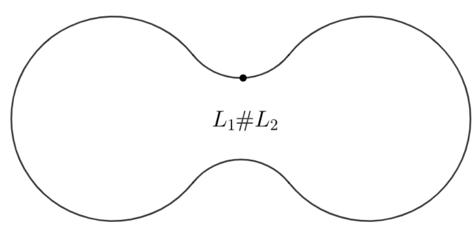
Proof. (1): For the module structure, we readily see that the generators for both sides are the same. Leibniz' rule is used for the differential (∂) of $C^k_{\mathrm{BN}}(L_1) \underset{\mathbb{F}[U]/U^k}{\otimes} C^k_{\mathrm{BN}}(L_2)$:

$$\partial^{X \otimes Y}(x, y) = (\partial^X x, y) + (x, \partial^Y y),$$

where $X = C^k_{\mathrm{BN}}(L_1)$, $Y = C^k_{\mathrm{BN}}(L_2)$, and x and y are respective elements.
(2): By Lemma 1, we may consider the following basepoint placement on L_1, L_2, and $L_1 \# L_2$ (basepoints are bolded dots, and the left side of L_1 and the right side of L_2 are not represented):



(a) $L_1 \cup L_2$



(b) $L_1 \# L_2$

Figure 2: Disjoint union and connected sum

Now, because we essentially omit the circles in the final resolution that contain the basepoint, we again have a correspondence between the generators for the two sides, and the argument is similar to the previous.

□

Let $-L$ be the link L but with all of the orientations reversed ([7]):

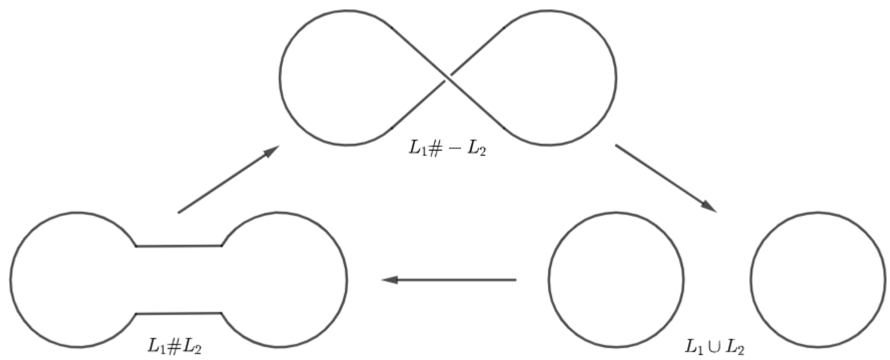


Figure 3: The exact triangle

Lemma 4. *There exists an exact triangle*

$$\ldots \to \mathrm{BN}^k(L_1\#L_2) \to \mathrm{BN}^k(L_1\#{-L_2}) \to \mathrm{BN}^k(L_1 \cup L_2) \to \mathrm{BN}^k(L_1\#L_2) \to \ldots.$$

Proof. Consider the resolution cube of the link $L_1\#{-L_2}$, which we will write as L. All resolutions of L contain either L_0 or L_1, which denote L with a 0 and 1 resolution at the intersecting crossing, respectively (we abuse notation: L_1 is not the same as in the statement of the lemma). The resolution cube can be divided accordingly into two parts, which we will call the resolution cube of L_0 and L_1, respectively. Note that all arrows in the resolution cube of L that go between the resolution cubes of L_0 and L_1 point from the resolutions in the cube of L_0 to resolutions in the cube of L_1. Thus the Bar-Natan chain complex of L_1 is a subcomplex of $C^k_{\mathrm{BN}}(L)$, and the chain complex of L_0 is the quotient complex $C^k_{\mathrm{BN}}(L_0) = C^k_{\mathrm{BN}}(L)/C^k_{\mathrm{BN}}(L_1)$. This gives us a short exact sequence:

$$0 \to C^k_{\mathrm{BN}}(L_1) \to C^k_{\mathrm{BN}}(L) \to C^k_{\mathrm{BN}}(L_0) \to 0.$$

Theorem 2.1 from [2] shows that the above induces the desired exact triangle in homologies. □

Lemma 5. *Given an exact triangle of vector spaces:*

$$\ldots \to A \xrightarrow{g} B \xrightarrow{f} C \xrightarrow{h} A \to \ldots,$$

we have the following:

$$\dim(A) = \dim(B) + \dim(C) - 2\dim(\mathrm{im}(f)).$$

Proof. By the rank-nullity formula and exactness, we have:

$$\begin{aligned}
\dim(A) &= \dim(\mathrm{im}\, g) + \dim(\ker g) \\
&= \dim(\ker f) + \dim(\mathrm{im}\, h) \\
&= \dim(B) - \dim(\mathrm{im}\, f) + \dim(C) - \dim(\ker h) \\
&= \dim(B) + \dim(C) - 2\dim(\mathrm{im}(f)).
\end{aligned}$$

$\square$

Recall that different orientations of different components on links results in many different choices of connected sum. However, the following is true:

Lemma 6. $\mathrm{BN}(L_1 \# L_2)$ *is independent of the choice of* $L_1 \# L_2$*, up to isomorphism.*

Proof. Part two of Lemma 3 states that $\widetilde{C}^k_{\mathrm{BN}}(L_1 \# L_2) \cong \widetilde{C}^k_{\mathrm{BN}}(L_1) \underset{\mathbb{F}[U]/U^k}{\otimes} \widetilde{C}^k_{\mathrm{BN}}(L_2)$. The Kunneth formula implies that the Bar-Natan homology of the right side is determined by $\widetilde{\mathrm{BN}}^k(L_1)$ and $\widetilde{\mathrm{BN}}^k(L_2)$. Thus, $\widetilde{\mathrm{BN}}^k(L_1 \# L_2)$ does not depend on the choice of connected sum. This result is extended to the unreduced Bar-Natan homology by Lemma 2, which implies that the dimension of the unreduced Bar-Natan homology as a vector space does not depend on the choice of connected sum. $\square$

Lemma 7. *The map in the exact triangle in Lemma 4 for* $\mathrm{BN}^k(L_1 \cup L_2) \to \mathrm{BN}^k(L_1 \# L_2)$ *is surjective.*

Proof. Consider the exact triangle in Lemma 4. By Lemma 6, we know that $\dim(\mathrm{BN}^k(L_1 \# -L_2)) = \dim(\mathrm{BN}^k(L_1 \# L_2))$, and from Lemma 3 we can deduce $\dim(\mathrm{BN}^k(L_1 \cup L_2)) = 2\dim(\mathrm{BN}^k(L_1 \# L_2))$. Now, by Lemma 5 applied to the exact triangle, dimension of the image of the considered map is $\dim(\mathrm{BN}^k(L_1 \# L_2))$, as desired. $\square$

Now we are ready to prove our main result:

Theorem 8. *For all* $k \geq 1$, $\dim(\mathrm{BN}^k(L))$ *is invariant under Conway mutation.*

Proof. We have the following diagram corresponding to the links in Figure 4 by Lemma 4, so that each of the three vertical columns and the three horizontal rows are exact sequences.

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \mathrm{BN}(L_{0,0}) \arrow[r, "f_0"] \arrow[d, "k_0"'] & \mathrm{BN}(L_{0,1}) \arrow[r] \arrow[d, "k_1"] & \mathrm{BN}(L_{0,\infty}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathrm{BN}(L_{1,0}) \arrow[r, "f_1"] \arrow[d] & \mathrm{BN}(L_{1,1}) \arrow[r] \arrow[d] & \mathrm{BN}(L_{1,\infty}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathrm{BN}(L_{\infty,0}) \arrow[r] \arrow[d] & \mathrm{BN}(L_{\infty,1}) \arrow[r] \arrow[d] & \mathrm{BN}(L_{\infty,\infty}) \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} &
\end{tikzcd}$$

Note that $L_{\infty,1}$ and $L_{1,\infty}$ are Conway mutants written in standard form (see Figure 1), and it suffices to prove that $\dim(\mathrm{BN}^k(L_{\infty,1})) = \dim(\mathrm{BN}^k(L_{1,\infty}))$. Lemma 7 implies f_0 and k_0 are

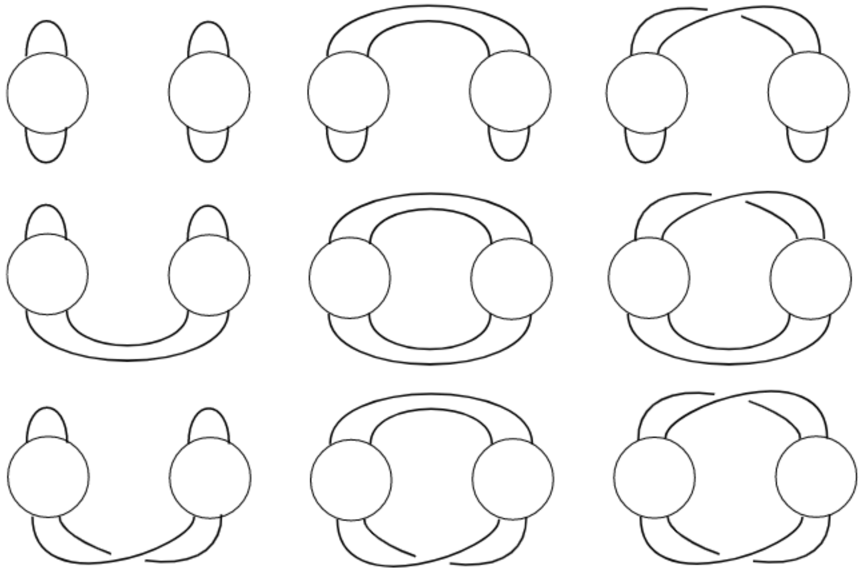


Figure 4: Links obtained from resolving two crossings in $L_{\infty,\infty}$ on the bottom right. Picture taken from [7].

surjective, and commutativity of the quotient map with merges and splits implies $f_1 \circ k_0 = k_1 \circ f_0$ (See [9]). We then have $\mathrm{im}(k_1) = \mathrm{im}(k_1 \circ f_0) = \mathrm{im}(f_1 \circ k_0) = \mathrm{im}(f_1)$. Lemma 6 implies $\dim(\mathrm{BN}^k(L_{0,1})) = \dim(\mathrm{BN}^k(L_{1,0}))$, so by Lemma 5:

$$\begin{aligned}
\dim(\mathrm{BN}^k(L_{1,\infty})) &= \dim(\mathrm{BN}^k(L_{1,1})) + \dim(\mathrm{BN}^k(L_{1,0})) - 2\dim(\mathrm{im}(f_1)) \\
&= \dim(\mathrm{BN}^k(L_{1,1})) + \dim(\mathrm{BN}^k(L_{0,1})) - 2\dim(\mathrm{im}(k_1)) \\
&= \dim(\mathrm{BN}^k(L_{\infty,1})).
\end{aligned}$$

□

At this point, it is natural to propose the following conjecture:

Conjecture 9. *The Bar-Natan homology $BN(L)$, as a module over $\mathbb{F}[U]$, is invariant under Conway Mutation. Similarly, the Bar-Natan homology $BN^m(L)$, as a module over $\mathbb{F}[U]/U^m$, is invariant under Conway Mutation.*

The main difficulty on the conjecture is due to a lack of Lemma 5 when working with modules (instead of vector spaces). Therefore, just controlling the dimension of the image of maps in the exact triangle is not enough to prove Conjecture 9, one needs to better understand these maps as maps between modules.

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