Enumerative Combinatorics

Nathaniel Liberman
Nhat Pham
Mentor: Svetlana Makarova

MIT PRIMES

December 15, 2017

Plan

Principle of Inclusion-Exclusion

Simple form

A well-known formula

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - \\
- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + \\
+ |A_1 \cap A_2 \cap A_3|
\end{aligned}
$$

Theorem

Given sets $A_1, A_2, ..., A_n$, we have the following formula for the number of elements in the union:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \leq i_1 < i_2 ... < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| \right).$$

Algebraic form

Theorem (Principle of Inclusion-Exclusion)

Let S be an set with n elements. Let V be the 2^n- dimensional vector space (over some field $\mathbb{K}$) of all functions $f : 2^S \to \mathbb{K}$. Let $\phi : V \to V$ be the linear transformation defined by:

$$\phi f(T) = \sum_{Y \supseteq T} f(Y), \forall T \subseteq S$$

Then $\forall T \subseteq S$:

$$\phi^{-1} f(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f(Y).$$

Applications

A typical setting

- A – a set of objects we study, *e.g. a set of humanoids*
- S – a set of interesting properties of the objects in a set A, *e.g. elf, religious, female*
- T – a subset of S, *e.g. is elf*
- $f_=(T)$ is the number of objects in A that have only the properties in the set T
- $f_\geq(T) = \phi(f_=(T)) = \sum_{Y \supseteq T} f_=(Y)$ is the number of objects in A that have at least the properties in the set T
- If we know $f_\geq(T)$, then we can compute $f_=(T)$ as: $f_=(T) = \left(\phi^{-1} f_\geq\right)(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f(Y)$

Example

Let us consider a fantasy town, and assume that there were two surveys.

Results of the first survey:

- 2100 female humanoids
- 950 human women and 900 female elves
- 1900 humans and 1850 elves

Results of the second survey:

- 1000 religious humanoids
- 200 religious humans and 500 religious elves
- 50 religious human women and 300 religious female elves

Question

How many non-religious male elves are there?

Example

Setting

$A = \{\text{all humanoids in town }\}$
$S = \{\text{female, elf, religious}\}$

Observation

Number of non-religious male elves is $f_=(\{\text{elf}\})$

Calculation

$$\begin{aligned} f_=(\{\text{elf}\}) = f_\geq(\{\text{elf}\}) - f_\geq(\{\text{female, elf}\}) - f_\geq(\{\text{religious, elf}\}) + \\ + f_\geq(\{\text{religious, female, elf}\}) = \\ = 1850 - 900 - 500 + 300 = 750 \end{aligned}$$

Generating functions

Introduction

Definitions

An *ordinary generating function* of a sequence $f(n)$ is a formal power series

$$F(x) = \sum_{n \geq 0} f(n) x^n,$$

while its *exponential generating function* is

$$G(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!}$$

Fundamental property of rational generating functions

Theorem

$$\text{Let} \quad \alpha_1, \alpha_2, ..., \alpha_d \in \mathbb{C}, \quad d \geq 1, \quad \text{and} \quad \alpha_d \neq 0$$

The following conditions on a function $f : \mathbb{N} \to \mathbb{C}$ are equivalent:

a.

$$\sum_{n \geq 0} f(n)x^n = \frac{P(x)}{Q(x)},$$

where $Q(x) = 1 + \alpha_1 x + \alpha_2 x^2 + ... + \alpha_d x^d$, and $P(x)$ is a polynomial in x of degree less than d.

b.

$\forall n \geq 0$:

$$f(n+d) + \alpha_1 f(n+d-1) + \alpha_2 f(n+d-2) + ... \alpha_d f(n) = 0$$

Generating function for Fibonacci sequence

Important Example

$$f(n) = F_n \quad \text{- Fibonacci sequence}$$

Compare $F_{n+2} - F_{n+1} - F_n = 0$ with statement b. to obtain from a.

$$\sum_{n\geq 0} F_n x^n = \frac{ax+b}{1-x-x^2}$$

and from initial conditions $F_0 = 0$ and $F_1 = 1$

$$\sum_{n\geq 0} F_n x^n = \frac{x}{1-x-x^2}$$

Explicit expression for Fibonacci numbers

Equivalently

$$\sum_{n\geq 0} F_n x^n = \frac{x}{(1-\varphi x)(1-\bar{\varphi}x)}$$

$$\text{with} \quad \varphi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \bar{\varphi} = \frac{1-\sqrt{5}}{2} = 1-\varphi = -\frac{1}{\varphi}$$

Hence as the Taylor series coefficients

$$F_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$

Alternating Permutations and Euler Numbers

Let $\mathfrak{S}_n$ be a set of permutations of $[n]$.
A permutation $w = w_1 w_2 ... w_n \in \mathfrak{S}_n$ is *alternating* if

$$w_1 > w_2 < w_3 > w_4 < ...$$

Definition

The number of *alternating permutations* $w \in \mathfrak{S}_n$ is called an *Euler number* E_n (with $E_0 = 1$).

Reverse Alternating Permutations

A permutation $w = w_1 w_2 ... w_n \in \mathfrak{S}_\mathfrak{n}$ is *reverse alternating* if

$$w_1 < w_2 > w_3 < w_4 > ...$$

Proposition

The number of *reverse alternating permutations* in $\mathfrak{S}_\mathfrak{n}$ is also E_n.

Proof

Since $w = w_1 w_2 ... w_n \in \mathfrak{S}_\mathfrak{n}$ is alternating if and only if

$$\tilde{w} = (n + 1 - w_1)\ (n + 1 - w_2)...\ (n + 1 - w_n)$$

is reverse alternating, there are as many reverse alternating as alternating permutations in $\mathfrak{S}_\mathfrak{n}$.

Generating Function for Euler Numbers

Theorem

The exponential generating function for Euler numbers is

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

Since $\sec x$ is an even function and $\tan x$ is odd, this is equivalent to

$$\sum_{n\geq 0} E_{2n} \frac{x^{2n}}{(2n)!} = \sec x$$

$$\sum_{n\geq 0} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \tan x$$

Proof

Let $S \subset [n]$ with $\#S = k$, and $\bar{S} = [n] \setminus S$. Choose reverse alternating permutations u of S and v of $\bar{S}$ in E_k and E_{n-k} ways. If $n \geq 1$,

$$w = u^r * (n+1) * v$$

uniquely represents every alternating and reverse alternating permutation of $[n+1]$. Hence

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}, \quad n \geq 1$$

For $G(x) = \sum_{n\geq 0} E_n \frac{x^n}{n!}$ with $E_0 = E_1 = 1$,

$$2G' = G^2 + 1, \quad G(0) = 1$$

$$G(x) = \sec x + \tan x$$