Mutli-Crossing Numbers for Knots

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What is this all about?

A fundamental problem in knot theory:

Determine whether two knots are **different**

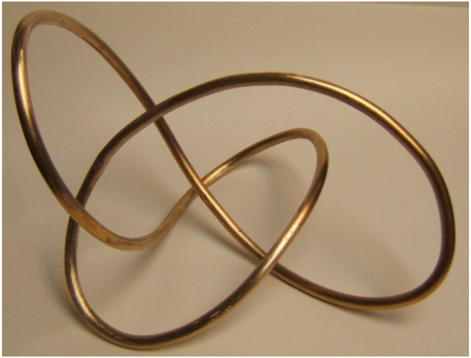
Applications to...

- **Biology**
  - Knotting in DNA
  - Molecular Knots
- **Physics**
  - Statistical Mechanics
  - Polymer Chains
- **Computer Science**
  - Deep Learning
  - Quantum Computers

What is a Knot?

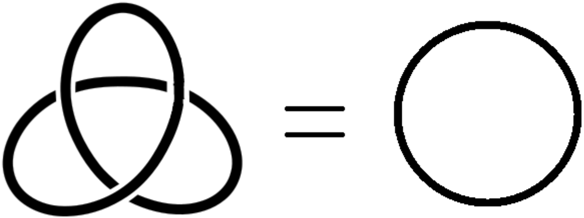
Definition

A knot is a closed curve in R^3 homeomorphic to a circle.



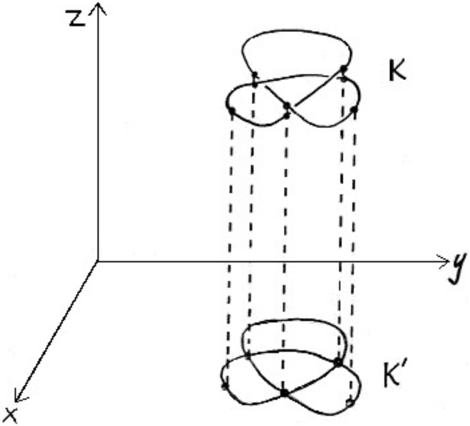
Equivalence of Knots

Two knots are the same if one can continuously deform one knot into the other knot.



Projection of a knot

The projection of a knot K onto K' on a plane. K' with the additional information of which strand is over and which is under at each crossing is called a knot diagram of K.



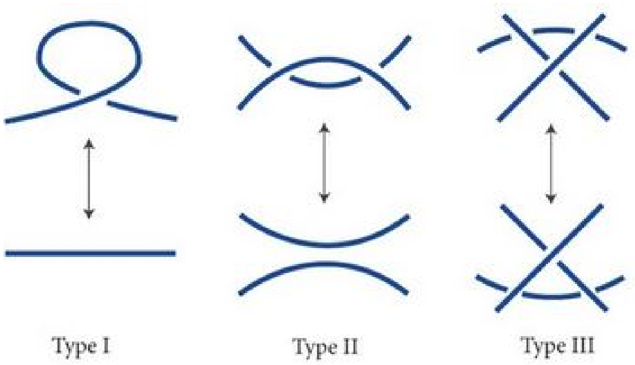
Knot diagram

Given two knot diagrams, how can we tell whether they represent the same knot or two different knots?

A partial answer was given by Reidemeister (1926):

- If two knot diagrams are connected by Reidemeister moves, they represent the same knot.

Reidemeister Moves



Reidemeister Theorem

Theorem (Reidemeister 1926)

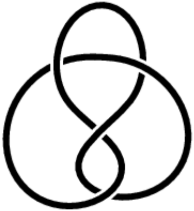
Two knot diagrams represent the same knot if and only if they are related by a finite sequence of Reidemeister moves.

Double crossing

Theorem

Any knot has a projection with a finite number of crossing and each crossing is a double crossing.

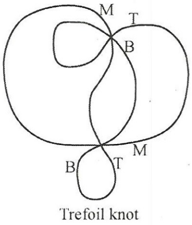
- *Crossing number* of a knot K, $c(K)$, is the least number of crossings that occur in any projection of K.



Multiple crossing

Theorem (Adams 2012)

Given any integer $n \geq 2$ and any knot, there exists a projection with only n-crossings.



n-crossing number $c_n(K)$

- An n-crossing projection of a knot K has n strands at each crossing.
- $c_n(K) = \min\{\#\text{ of } n\text{-crossings in a } n\text{-crossing projection of } K\}$

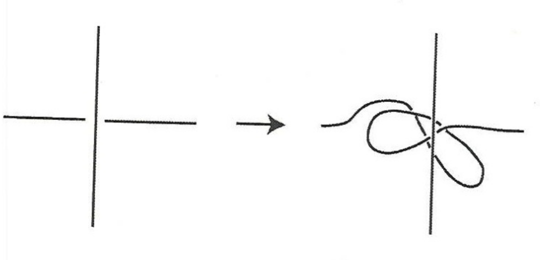
$c_{2k}(K)$ and $c_{2k+1}(K)$ are decreasing

Theorem (Adams 2012)

For any integer $n \geq 2$ and any knot K, $c_n(K) \geq c_{n+2}(K)$, i.e.,

$$c_2(K) \geq c_4(K) \geq c_6(K) \geq \cdots$$

$$c_3(K) \geq c_5(K) \geq c_7(K) \geq \cdots$$



$c_2(K)$ and $c_3(K)$

Theorem (Adams, 2012)

For any knot K,

$$\frac{c_2(K)}{3} \leq c_3(K) \leq c_2(K) - 1.$$

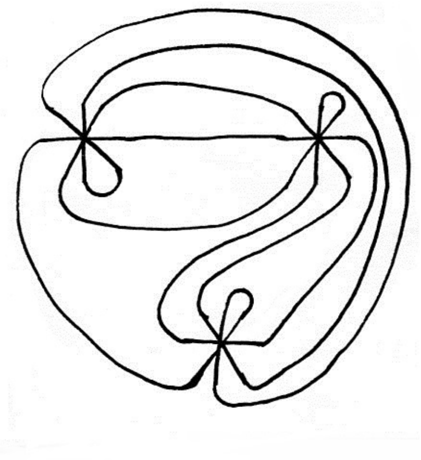
Multiple crossing number

Theorem (T., 2017)

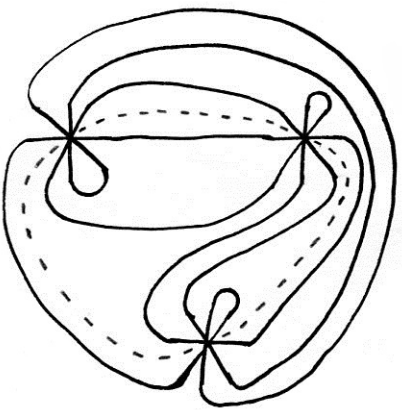
For any positive even integer n and any knot K,

$$c_n(K) \geq c_{2n-1}(K).$$

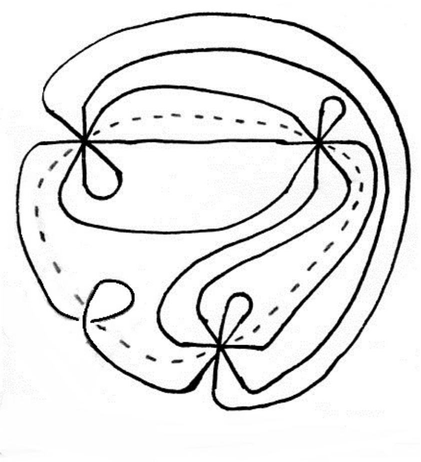
Sketch of proof 1



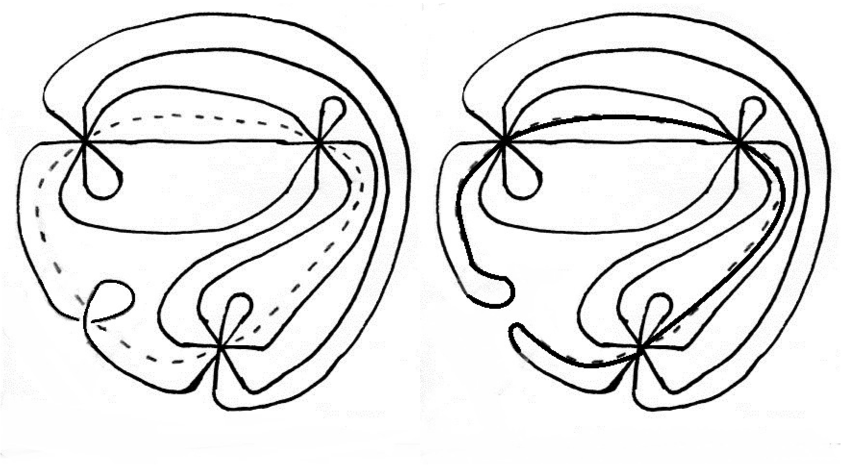
Sketch of proof 2



Sketch of proof 3



Sketch of proof 4



Summary

- Knots and their diagrams
- Multi-crossing in a knot diagram
- Define multi-crossing number $c_n(K)$
- New result $c_{2m}(K) \geq c_{4m-1}(K)$

Directions for future research

- Prove that $c_n(K) \geq c_{n+1}(K)$.
- Prove strict inequalities between crossing numbers.
- Find lower bounds for petal number and übercrossing number.

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References

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[3] C. Adams, J. Hoste, and M. Palmer, *Diagramatic Moves for 3-Crossing Knot Diagrams*, preprint.

[4] P. Cromwell, *Knots and Links*, Cambridge University Press, 2004