Second Gonality of Erdős-Rényi Random Graphs

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Research Problem

Problem

What is the asymptotic behaviour of the expected value of the second gonality of an Erdős-Rényi Random Graph $G(n, p)$?

Motivation and Context

Chip-Firing Game

Divisors on graphs was inspired by a "chip-firing game" where values at each vertex are thought of as a pile of chips. Chips would be moved from pile to pile by the rules of chip-firing.

Winning the Game

A vertex with a negative number of chips in its pile is considered to be"in debt". A divisor is winning iff it is equivalent to a divisor with no vertices in debt.

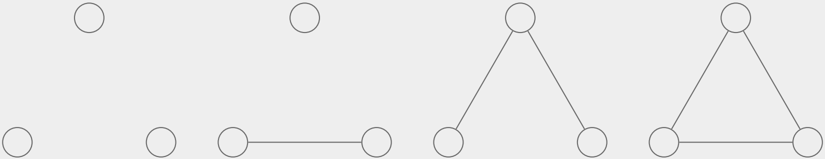
k-th gonality is the minimum number of chips necessary on the graph to ensure that an "opponent" cannot make the divisor a losing divisor by taking away any k chips from the divisor.

Preliminaries

Definition

For $0 < p < 1$, an Erdős-Rényi Random Graph $G(n, p)$ is a simple graph with n vertices and an edge between any two distinct vertices with probability p.

Example: 3 Vertices



$(1-p)^3$ $\quad$ $3p(1-p)^2$ $\quad$ $3p^2(1-p)$ $\quad$ p^3

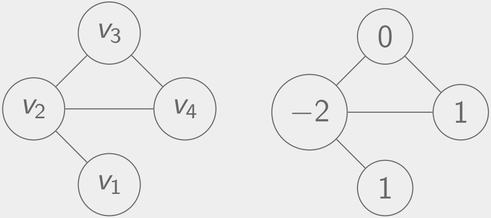
Preliminaries, cont.

Definition

A divisor D on a graph G is a formal $\mathbb{Z}$-linear combination of the vertices of G,

$$D = \sum_{v \in V(G)} D(v)v.$$

Example



$D = v_1 - 2v_2 + v_4$

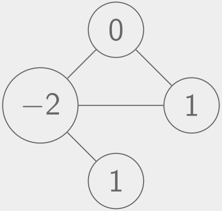
Preliminaries, cont.

Definition

Degree of a divisor D is the sum of the coefficients at each vertex,

$$\deg(D) = \sum_{v \in V(G)} D(v).$$

Example



Degree of this divisor is 0.

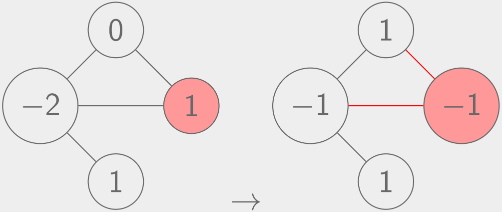
Preliminaries, cont.

Definition

A specific vertex is "fired" (in a process called "chip-firing") by transferring exactly one value along each connected edge to each directly adjacent vertex.

Note that degree is invariant under chip-firing.

Example

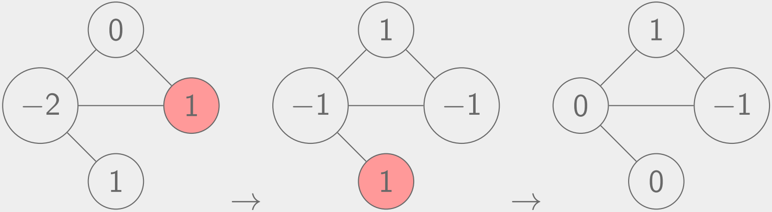


Preliminaries, cont.

Definition

Two divisors on a graph are said to be equivalent if one can be obtained from the other through a series of chip-firing moves.

Example



Any of these three divisors are pairwise equivalent.

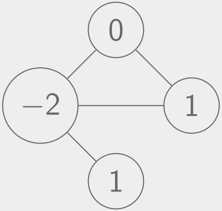
Preliminaries, cont.

Definition

A divisor D is said to be effective if there are a non-negative number of chips on all vertices of its associated graph, or

$$v \in V(G) \implies D(v) \geq 0.$$

Example



This divisor is not effective because one of its nodes has a negative coefficient.

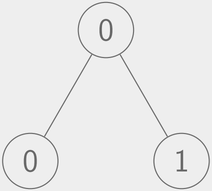
Preliminaries, cont.

Definition

A divisor D has rank r if r is the largest integer such that for every effective divisor E with degree r, $D - E$ is equivalent to an effective divisor. Note that if a divisor D has degree less than 0 it is defined to have a rank of -1. Also, note that a divisor D must have degree greater than or equal to its rank.

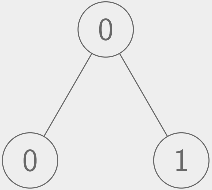
Example

This divisor has rank 1.



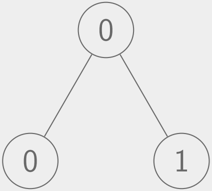
Preliminaries, cont.

The rank is at most 1 because rank is at most degree. The rank is at least 1 as for each effective divisor E of degree 1 satisfies $D - E$ is equivalent to an effective divisor. Thus, the rank is 1. Every possible $D - E$ will be shown to be equivalent to an effective divisor on the following slides.

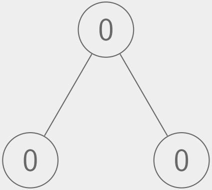


Preliminaries, cont.

Effective divisor E:

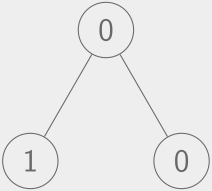


$D - E$:

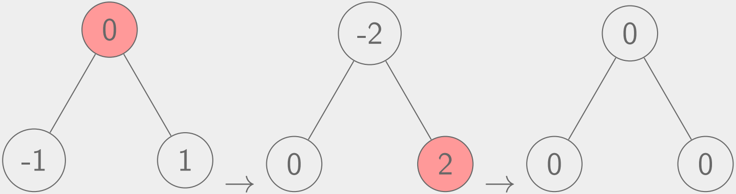


Preliminaries, cont.

Effective divisor E:

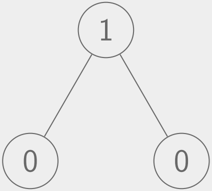


$D - E$:

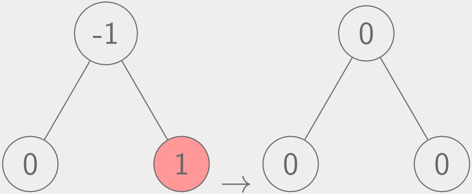


Preliminaries, cont.

Effective divisor E:



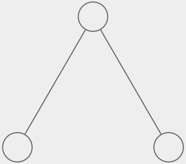
$D - E$:



Definition

Given a fixed graph G, the k-th gonality of G is the minimum degree for a divisor on G to have rank k.

Example



The first gonality of this graph is 1.

First Gonality

Theorem (Deveau et al.)

Let $p(n) = \frac{c(n)}{n}$, and suppose that $c(n) \ll n$ is unbounded. Then

$$\mathbb{E}(\text{gon}\, G(n,p)) \sim n.$$

Results

Computations

Let $F_n(p) = \mathbb{E}(\text{gon}_2\, G(n,p))/n$, then we have the following results:

$$
\begin{aligned}
F_1(p) =& 2 \\
F_2(p) =& 2 - p \\
F_3(p) =& 2 - 2p + p^3 \\
F_4(p) =& 2 - 3p + 3p^3 + 2.25p^4 - 4.5p^5 + 1.25p^6 \\
F_5(p) =& 2 - 4p + 6p^3 + 9p^4 - 10.8p^5 - 37p^6 + 58p^7 - 6p^8 \\
& - 30p^9 + 13.8p^{10}
\end{aligned}
$$

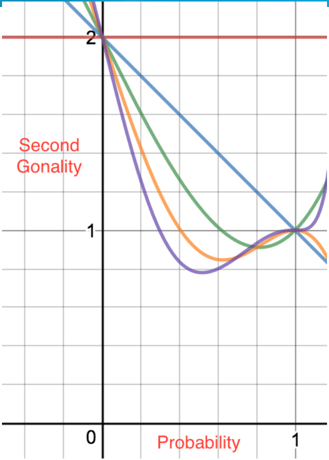


Figure: F_1 F_2 F_3 F_4 F_5

An Explicit Bound on the Second Gonality

Theorem

The second gonality of an Erdős-Rényi Random Graph is bounded above by

$$\mathbb{E}(\operatorname{gon}_2 G(n,p)) \leq n(1 + e^{-c(n)}).$$

Corollary

$$\frac{\mathbb{E}(\operatorname{gon}_2 G(n,p))}{n} \sim 1.$$

for $c(n) \to \infty$.

Proof

Proof.

Call a vertex *isolated* if it has no neighbor. Consider the divisor D with two chips on each isolated vertex and one chip on all other vertex. Any divisor E with two chips on different vertices trivially satisfies $D - E$ effective, whereas if both chips of E are on a vertex v, then firing all other vertices in divisor $D - E$ leaves an effective result.

Thus, the expected gonality is bounded above by $n + k$ where k is the expected number of isolated vertices. The probability any given vertex is isolated is $(1-p)^{n-1}$ and thus the expected number of isolated vertices is $k = n(1-p)^{n-1} = n(1 - \frac{c(n)}{n})^{n-1}$, approaching $ne^{-c(n)}$ as n tends to infinity. Hence our upper bound is $n(1 + e^{-c(n)})$. □

Future Research

- Look for a conclusive bound to show that
$$\mathbb{E}(g_2(G(n,p))) \sim n.$$
.
- Try to find another approach to bounding the second gonality that can generalize to higher cases.

References

- Matthew Baker and Serguei Norine *Riemann-Roch and Abel-Jacobi theory on a finite graph* https://arxiv.org/abs/math/0608360
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- Andrew Deveau and David Jensen and Jenna Kainic and Dan Mitropolsky *Gonality of Random Graphs* 2014, https://arxiv.org/abs/1409.5688
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