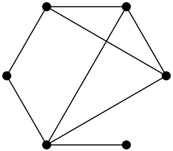
Classifying Graph Lie Algebras

Michael Gintz
Mentor: Dr. Tanya Khovanova

PRIMES Conference, May 20 2017

Defining a Graph



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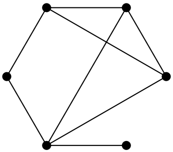

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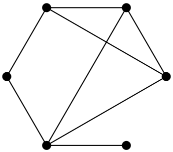

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There are many well-known graph theory problems:

- The Konigsberg Bridges Problem
- The Traveling Salesman
- The Four-Color Theorem

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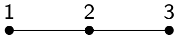
- $e_i^2 = -1$.
- e_i and e_j anticommute ($e_i e_j = -e_j e_i$) when vertices i and j are connected: otherwise they commute ($e_i e_j = e_j e_i$).

Manipulating Monomials



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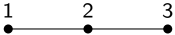

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For a set $\alpha = \{e_{i_1}, ..., e_{i_k}\}$, the monomial e_α is equal to $e_{i_1}...e_{i_k}$.

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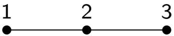

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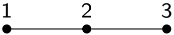

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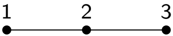

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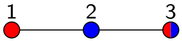

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Theorem

Monomials e_α and e_β anticommute if there exist an odd number of pairs of connected vertices with one in α and one in β.

Structure of a Graph Algebra

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The *symmetric difference* of two sets α, β is defined as follows:

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Lemma

For two sets α, β, the product $e_\alpha e_\beta$ is equal to $\pm e_{\alpha \triangle \beta}$.

Mutiplying $e_1 e_3 \cdot e_2 e_3$ always yields $\pm e_1 e_2$:

$$\begin{array}{ll} e_1 & \cancel{e_3} \\ \quad e_2 & \cancel{e_3} = \pm e_1 e_2. \end{array}$$

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- It must be alternative: $[x, x] = 0$.
- It must satisfy the Jacobi Identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

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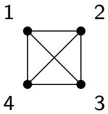
- Every generator $e_1, ..., e_n$ is in $\mathfrak{L}(G)$.
- No Lie subalgebra of $\mathfrak{L}(G)$ contains every generator.

Forming a Lie Algebra



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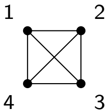

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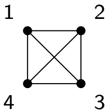

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We can combine each pair of generators using the Lie bracket:
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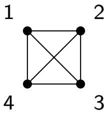

We can form the Lie algebra of the complete graph K_4 by hand.
We must have monomials e_1, e_2, e_3, e_4.
We can combine each pair of generators using the Lie bracket:
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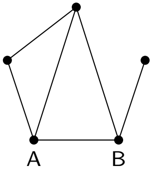
We can not have any more monomials, as every pair of monomials other than pairs of generators contain commuting monomials. Therefore our Lie Algebra has 10 dimensions.

Swapping our Graph



We will create a series of alterations on our graph called a *swap* about vertex A with respect to B. This is denoted as ${}_AG_B$. We consider all vertices $v \neq A, B$ connected to A.

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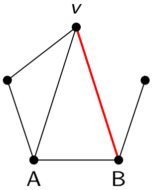

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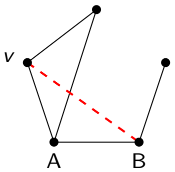

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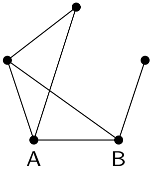

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Theorems Concerning Swaps

Swapping our graph always preserves its algebra:

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For all graphs G, algebras $\mathcal{A}(G)$ and $\mathcal{A}({}_AG_B)$ are isomorphic.

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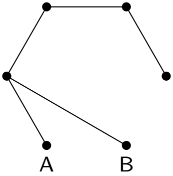
Sometimes, swapping our graph can preserve the Lie algebra as well:

Theorem

For all graphs G, Lie algebras $\mathfrak{L}(G)$ and $\mathfrak{L}({}_AG_B)$ are isomorphic when A and B are connected.

Removing Leaves from our Graph



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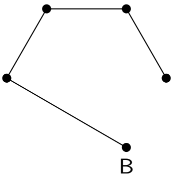

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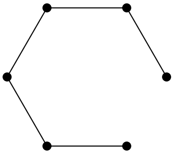
Theorem

$$\mathfrak{L}(G) = \mathfrak{L}(G \backslash A) \oplus \mathfrak{L}(G \backslash A).$$

Completely Classified Graphs



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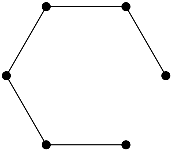

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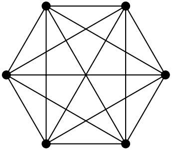
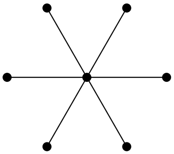


The Lie algebra of a complete graph with n vertices is isomorphic to that of a path graph with n vertices.

More Completely Classified Graphs



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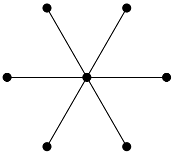

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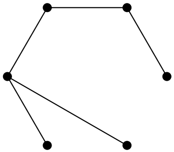

The Lie algebra of a $n-2$ vertex graph with 2 leaves attached to the same end (Dynkin diagram D_n) is a direct sum of two copies of the Lie algebra of an $n-1$ vertex path graph.

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- Generalize our decomposition move.
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- Relate graph Lie algebras to matrix algebras.

Acknowledgements

- Dr. Tanya Khovanova
- The PRIMES Program
- My parents

Questions?