Coin Games and 5-way Scales

Joshua Lee
Mentor: Dr. Tanya Khovanova

PRIMES Conference

May 20 2017

Three Way Scale

- N total identical-looking coins
- Balance scale
- 1 fake, lighter than real
- Goal: determine fake coin using the least amount of weighings as possible.

Solution

Information Theoretical Bound

Each weighing gives three possible outcomes. A certain sequence of outcomes must point to a unique coin, so the number of these sequences is at least the number of the coins.

Solution

Information Theoretical Bound

Each weighing gives three possible outcomes. A certain sequence of outcomes must point to a unique coin, so the number of these sequences is at least the number of the coins.

Strategy

Divide the pile into three equal piles and weigh any two of the piles. This must allow us to determine which pile the fake coin is contained in.

Solution

Information Theoretical Bound

Each weighing gives three possible outcomes. A certain sequence of outcomes must point to a unique coin, so the number of these sequences is at least the number of the coins.

Strategy

Divide the pile into three equal piles and weigh any two of the piles. This must allow us to determine which pile the fake coin is contained in.

- $w = \lceil \log_3 N \rceil$

Five Way Scale

- Two more possible outcomes
- $d = \#$ of fake coins on left pan - $\#$ of fake coins on right pan

MUCH LESS	LESS	EQUAL	MORE	MUCH MORE
$d \geq 2$	$d = 1$	$d = 0$	$d = -1$	$d \leq -2$

Five Way Scale

- Two more possible outcomes
- $d = \#$ of fake coins on left pan - $\#$ of fake coins on right pan

MUCH LESS	LESS	EQUAL	MORE	MUCH MORE
$d \geq 2$	$d = 1$	$d = 0$	$d = -1$	$d \leq -2$

- 2 fake coins

Information Theoretical Bound

- $5^w \geq \binom{N}{2}$

Information Theoretical Bound

- $5^w \geq \binom{N}{2}$
- "Equal" is the outcome with most remaining possibilities

Lemma

After one weighing, let a_1,a_2,a_3,a_4,a_5 be the number of remaining possibilities of the fake coins for the outcomes MUCH LESS, LESS, EQUAL, MORE, MUCH MORE respectively. Then,

$$max(a_1, a_2, a_3, a_4, a_5) = a_3$$

regardless of how many coins were on each pan.

Information Theoretical Bound

- $5^w \geq \binom{N}{2}$
- "Equal" is the outcome with most remaining possibilities

Lemma

After one weighing, let a_1,a_2,a_3,a_4,a_5 be the number of remaining possibilities of the fake coins for the outcomes MUCH LESS, LESS, EQUAL, MORE, MUCH MORE respectively. Then,

$$max(a_1, a_2, a_3, a_4, a_5) = a_3$$

regardless of how many coins were on each pan.

- $\frac{3}{5} \cdot 5^w \geq \binom{N}{2}$

Strategy

- Linear Strategy — Compare 2 coins with 2 coins: $w = \left\lfloor \frac{N}{2} \right\rfloor$

Strategy

- Linear Strategy — Compare 2 coins with 2 coins: $w = \left\lfloor \frac{N}{2} \right\rfloor$
- Better Strategy — Divide into 3 equal piles
  1. MUCH LESS: Problem Reduced to $\frac{N}{3}$
  2. LESS: $w = \log_2 N$
  3. EQUAL: One more weighing to reduce problem to $\frac{N}{3}$
- $w = 2\lceil \log_3 \frac{N}{3} \rceil$ weighings

Further Research

- Better Strategy

Further Research

- Better Strategy
- Improved Bounds

Conjecture

If w denotes the maximum number of weighings in any strategy that guarantees finding the fake coins, and N is the total number of coins, then there exists a constan k such that

$$k \cdot 3^w \geq \binom{N}{2}$$

for all N.

Further Research

- Better Strategy
- Improved Bounds

Conjecture

If w denotes the maximum number of weighings in any strategy that guarantees finding the fake coins, and N is the total number of coins, then there exists a constan k such that

$$k \cdot 3^w \geq \binom{N}{2}$$

for all N.

- General n-way Scale

Game Theory

- Nim — Basic Game

- Goal: Take the last stone

- Nim — Basic Game

- Goal: Take the last stone
- Winning (P) positions and losing (N) positions

Minimum Excluded

Definition

The **minimum excluded value** (often shorted as **mex**) of a subset of some well-ordered set is the smallest value not included in the set.

For our use, we will assume that we are using set of non-negative integers.

Example

- $\text{mex}(0, 1, 3) = 2$
- $\text{mex}(1, 2, 3) = 0$
- $\text{mex}(0, 2, 4, 6...) = 1$

Grundy Numbers

Sprague-Grundy Theorem

Every impartial game is equivalent to a nim-heap of a certain size.

- Game equivalent to the number of stones in Nim
- mex of the set of reachable Grundy Numbers

Disclaimer

The following research was begun by the following people:

- Kyle Burke
- Tanya Khovanova
- Richard J. Nowakowski
- Amelia Rowland
- Craig Tennenhouse

Aequitas

- Aequitas — Latin concept of equity
- Game regarding the classic coin problem
- Must reveal information every turn
- Observer cannot know the fake coin
- Player loses if there is no legal move

Grundy Numbers for Aequitas

- One Final position - 2 remaining possible coins
- P-position - final position
- N-positions - every other position

Game Values

N	Grundy Number
$4k+3$	$2k$
$4k+4$	$2k+1$
$4k+5$	$2k+1$
$4k+6$	$2k+1$

Modified Aequitas; Game 2

- Fake coin either heavier or lighter
- Observer cannot know fake coin
- One Final position — only P-position

Game Values

N	Grundy Number
$2k$	$2k-2$
$2k+1$	1

Modified Aequitas; Game 3

- Observer cannot know relative weight of fake coin
- Two final positions - only P-positions
- Equal Grundy numbers as Game 2

Game Values

N	Grundy Number
$2k$	$2k-2$
$2k+1$	1

Future Research

- Limit to number of coins on each scale
- Other games

Acknowledgements

I would like to thank Dr. Khovanova and the PRIMES-USA program for making this research possible, as well as my parents for supporting me from the start.