The PRIMES 2014 problem set

Dear PRIMES applicant!

This is the PRIMES 2014 problem set. Please send us your solutions as part of your PRIMES application by December 1, 2014. For complete rules, see http://web.mit.edu/primes/apply.shtml

Note that this set contains three sections: "General math problems" (for all tracks) and two sections corresponding to the three research tracks of PRIMES 2014 ("Advanced math" and "Computer science"). Please solve as many problems as you can in the General math section, and also in the section(s) corresponding to the track(s) for which you are applying (Computer Science and Computational biology applicants should solve the problems from the Computer Science section, in additional to "General math").

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF (preferred), DOC, or JPG file. The name of the attached file must start with your last name, for example, "smith-solutions." Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people's help.* This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that posting these problems on problem-solving websites before the application deadline is not allowed. Applicants who do so will be disqualified and may face integrity issues.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day, and think about them, on and off, over some time (several days). We encourage you to apply if you can solve at least 50% of the problems. [1]

Enjoy!

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[1]We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

General math problems

Problem G1. Let $n > 2$ be an integer. Find explicitly a nonzero polynomial P of degree $2n$ with integer coefficients and leading coefficient 1 such that $P(2^{1/2} + 2^{1/n}) = 0$. How many real roots does P have?

Solution Let $x = 2^{1/2} + 2^{1/n}$. Then $2^{1/n} = x - \sqrt{2}$. So

$$(x - \sqrt{2})^n = 2.$$

Thus

$$((x - \sqrt{2})^n - 2)((x + \sqrt{2})^n - 2) = 0.$$

So we can take

$$P(x) = (x^2 - 2)^n - 4 \sum_k \binom{n}{2k} 2^k x^{n-2k} + 4.$$

It has 4 real roots if n is even and 2 real roots if n is odd.

Problem G2. Each minute, a drunkard walks one step to the right with probability $1/3$, two steps to the right with probability $1/3$, and one step to the left with probability $1/3$. He is initially k steps to the right from a cliff. What is the probability that he will fall off? (If he is 0 steps away, he falls off.)

Solution. Let q_k be this probability. Then $q_k = (q_{k+2} + q_{k+1} + q_{k-1})/3$, where $q_0 = 1$. Also $q_k = q^k$, where $q = q_1 < 1$. So $3q = q^3 + q^2 + 1$, which implies that $q = \sqrt{2} - 1$. So the probability to fall off is $(\sqrt{2} - 1)^k$.

Problem G3. Does the equation

$$x^3 + 2x - y^2 = 1$$

have integer solutions?

Solution. No. $x^3 + 2x$ is divisible by 3, so $y^2 = -1$ modulo 3, which is impossible.

Problem G4. John's secret number is between 1 and 2^{16}, and you can ask him "yes or no" questions, but he may lie in response to one of the questions. Explain how to determine his number in 21 questions.

Solution. Question n, $1 \le n \le 15$: Is the n-th binary digit of X zero? Question 16. Have you lied on Questions 1-15? If "no", it means he did not lie (since he cannot lie twice), So we know the first 15 digits. Then ask about the last digit three times, and the answer given at least two times is correct, so we know the number.

If "yes" to question 16, then he lied on one of the questions 1-16. Use binary search (4 questions) to determine where he lied. This will tell us the first 15 digits, and we can use the 21st question (to which he has to tell the truth since he already lied) to find the last digit.

Problem G5. Let n be a fixed positive integer, and r a fixed positive number. Show that the number of positive integer solutions of the equation

$$\frac{1}{x_1} + \ldots + \frac{1}{x_n} = r$$

is finite.

Hint. Consider the smallest of the x_i.

Solution. The proof is by induction in n. The case $n = 1$ is clear. Let x_n be the smallest of the x_i. Then $x_n \le r/n$, so there are finitely many choices for x_1. Now fix x_n to be one of these values. We now need to solve

$$\frac{1}{x_1} + \ldots + \frac{1}{x_{n-1}} = r - \frac{1}{x_n},$$

and there are finitely many solutions by the induction assumption.

Problem G6. A round table has n seats. How many ways are there to seat k people at this table, so that no two of them sit next to each other? (two seatings are viewed as the same if each person sits on the same chair under both seatings).

Solution. There are n positions for the first person, and $(k-1)!$ cyclic orderings. Given these, the gaps between the people are g_i such that $g_i \ge 2$ and $g_1 + \ldots + g_k = n$. Let $a_i = g_i - 1$. Then $a_i + \ldots + a_k = n - k$, and $a_i \ge 1$. The number of solutions is $\binom{n-k-1}{k-1}$. Thus the number of arrangements is the product of all these, i.e. $N = n(n-k-1)\ldots(n-2k+1)$.

Problem G7. On a round table of diameter 2 feet there are 132 coins of diameter 1 inch. Show that one can put one more such coin on the table without overlap with the other coins.

Solution. Suppose that this cannot be done, and the number of coins is N. Enlarge each coin twice in diameter (keeping the same center). These enlarged coins must completely cover the table with the exception of the strip of width $1/2$ inch along the edge. So the area bound gives $N \ge 11.5^2 = 132.25$.

Advanced math problems

Problem M1.

A population starts with one amoeba. In each generation, each amoeba either divides in two, with probability $1/2$, or dies, with probability $1/2$. Let p_n be the probability that the population will survive n generations.

(a) What is p_4? (as a fraction).

(b) Find the limit $a = \lim_{n \to \infty} np_n$.

(c) Show that $p_n = \frac{a}{n} + \frac{b \log(n)}{n^2} + O(\frac{1}{n^2})$ as $n \to \infty$, and find b.

Solution (sketch). (a) We have $p_0 = 1$, and

$$1 - p_{n+1} = \frac{1}{2}((1 - p_n)^2 + 1),$$

so

$$p_{n+1} = p_n - \frac{p_n^2}{2}.$$

Thus $p_4 = 8463/32768$.

(b) To find asymptotics of this sequence, consider the sequence $x_n = C/n$. Then $n = C/x_n$, so $x_{n+1} = C/(n+1) = C/(C/x_n + 1) = x_n/(1 + x_n/C) = x_n - x_n^2/C + ...$ Thus, we see that our sequence behaves like x_n for $C = 2$, so p_n is equivalent to $2/n$, and the limit equals $a = 2$.

(c) To find b, substitute the expression of (c) into the recursion. This gives $b = -2$.

Problem M2. Let $V(n)$ be the vector space over the field $\mathbb{Q}$ of rational numbers spanned by the numbers $\log(2), \log(3), ..., \log(n)$. What is the dimension of $V(100)$? of $V(n)$ for a general n?

Solution. The fundamental theorem of arithmetic implies that a basis of $V(n)$ is formed by $\log(p)$, where p runs over primes between 1 and n. So $\dim V(n)$ is the number of primes between 1 and n, and $\dim V(100) = 25$.

Problem M3. Compute the limit

$$L = \lim_{n \to \infty} \left(\prod_{k=1}^{n-1} \cos \frac{\pi k}{2n} \right)^{1/n}.$$

Solution. We have

$$\log L = \frac{2}{\pi} \int_0^{\pi/2} \log(\cos u) du = \log(1/2),$$

so $L = 1/2$.

Problem M4. Three points A, B, C are chosen at random (with uniform distribution) and independently inside a circle with center O.

(a) What is the probability q that O is inside the triangle ABC?

(b) What is the probability p that C is inside the triangle OAB?

(c) What is the probability r that the points O, A, B, C are vertices of a convex 4-gon?

Solution. (a) Without loss of generality we may assume that the angle coordinate of A is zero. Assume that the angle coordinate of B is θ. We may assume that θ is between 0 and π. Then for O to be inside ABC, the angle coordinate of C should be between π and $\pi + \theta$. So

$$q = \frac{1}{\pi} \int_0^\pi \frac{\theta}{2\pi} d\theta = \frac{1}{4}.$$

(b) Assume the circle has unit radius. The chance is $p = \overline{S}/\pi$, where $\overline{S}$ is the average area of OAB. We may assume that A lies on the positive half-axis. If the polar coordinates of A are $(r_1, 0)$ and of B are (r_2, θ) then the area of OAB is

$$S = \frac{1}{2} r_1 r_2 |\sin \theta|.$$

So

$$\overline{S} = \frac{1}{\pi} \int_0^1 r_1^2 dr_1 \int_0^1 r_2^2 dr_2 \int_0^{2\pi} |\sin \theta| d\theta =$$

$$\frac{1}{9\pi} \int_0^{2\pi} |\sin \theta| d\theta = \frac{4}{9\pi}.$$

So $p = 4/9\pi^2$.

(c) It is $r = 1 - q - 3p = \frac{3}{4} - \frac{4}{3\pi^2}$.

Problem M5. (a) Find the number of matrices of size n by n over the field of two elements which are conjugate to a diagonal matrix. What is the answer for $n = 4$?

(b) What is the number of n by n matrices conjugate to a diagonal matrix over any finite field F_q?

Solution. (a) Such a matrix is a projection of some rank k. The centralizer of a projection of rank k is $GL_k(F_2) \times GL_{n-k}(F_2)$. So the desired number is

$$N(n) = |GL_n(F_2)| \sum_{k=0}^n |GL_k(F_2) \times GL_{n-k}(F_2)|^{-1} = \sum_{k=0}^n 2^{k(n-k)} \binom{n}{k}_2,$$

where $\binom{n}{k}_2 := \frac{(2^n-1)\ldots(2^{n-k+1}-1)}{(2^k-1)\ldots(2^1-1)}$. So $N(4) = 1+8\cdot15+16\cdot35+8\cdot15+1 = 802$.

(b) The answer is

$$N = \sum_{k_1+\ldots+k_q=n, k_i \geq 0} q^{\sum_{i<j} k_i k_j} \frac{n!_q}{k_1!_q \ldots k_q!_q},$$

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where $n!_q = (1 + q)(1 + q + q^2)...(1 + q + q^2 + ... + q^{n-1})$.

Problem M6. At a banquet, a round table is served for an even number n of guests, and there are cards on the plates with guest names. However, the guests seat themselves without paying attention to the cards.

(a) Show that the table can be rotated so that at least two guests are seated according to the cards.

(b) Show that if n is any odd number, then statement (a) is false.

Solution. (a) Let a_i be the assigned seat of the guest sitting at the i-th position (both i and a_i are residues modulo n). If $a_i - a_j = i - j$ for some i, j, then rotating the table by $(i - a_i)/n$ of the full circle, we will get the guests i and j to sit at their assigned seats. So let's assume that $a_i - a_j \neq i - j$ for any $i \neq j$. Then $a_i - i \neq a_j - j$, so the function $i \mapsto a_i - i$ must take all the possible values $0, 1, ..., n - 1$. But $\sum_i (a_i - i) = 0$, as a_i also take all possible values. So we would get $0 + 1 + ... + (n - 1) = n(n - 1)/2 = 0$ modulo n. But this holds only for odd n.

(b) One can define $a_i = 2i$, then $a_i - i = i$ and (a) is false.