The PRIMES-USA 2014 Problem Set

Dear PRIMES-USA applicant!

This is the PRIMES-USA 2014 Problem Set. Please send us your solutions as part of your application by November 15, 2013. For complete rules, see http://web.mit.edu/primes/PrimesUSAapply.shtml

Note that this set contains two parts: "General Math problems" and "Advanced Math." Please solve as many problems as you can in both parts.

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF (preferred), DOC, or JPG file. The name of the attached file must start with your last name, for example, "smith-solutions." Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that posting these problems on problem-solving websites before the application deadline is not allowed. Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 60% of the problems. [1]
Enjoy!

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[1] We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

General math problems

Problem G1. Let $n > 2$ be an integer. Find explicitly a nonzero polynomial P of degree $2n$ with integer coefficients and leading coefficient 1 such that $P(2^{1/2} + 2^{1/n}) = 0$. How many real roots does P have?

Problem G2. Each minute, a drunkard walks one step to the right with probability $1/3$, two steps to the right with probability $1/3$, and one step to the left with probability $1/3$. He is initially k steps to the right from a cliff. What is the probability that he will fall off? (If he is 0 steps away from the cliff, he falls off.)

Problem G3. Does the equation

$$x^3 + 2x - y^2 = 1$$

have integer solutions?

Problem G4. John's secret number is between 1 and 2^{16}, and you can ask him "yes or no" questions, but he may lie in response to one of the questions. Explain how to determine his number in 21 questions.

Problem G5. Let n be a fixed positive integer, and r a fixed positive number. Show that the number of positive integer solutions of the equation

$$\frac{1}{x_1} + ... + \frac{1}{x_n} = r$$

is finite.

Hint. Consider the smallest of the x_i.

Problem G6. A round table has n seats. How many ways are there to seat k people at this table, so that no two of them sit next to each other? (two seatings are viewed as the same if each person sits on the same chair under both seatings).

Problem G7. On a round table of diameter 2 feet there are 132 coins of diameter 1 inch. Show that one can put one more such coin on the table without overlap with the other coins.

Advanced math problems

Problem M1.

A population starts with one amoeba. In each generation, each amoeba either divides in two, with probability $1/2$, or dies, with probability $1/2$. Let p_n be the probability that the population will survive n generations.

(a) What is p_4? (as a fraction).

(b) Find the limit $a = \lim_{n \to \infty} np_n$.

(c) Show that $p_n = \frac{a}{n} + \frac{b \log(n)}{n^2} + O(\frac{1}{n^2})$ as $n \to \infty$, and find b.

Problem M2. Let $V(n)$ be the vector space over the field $\mathbb{Q}$ of rational numbers spanned by the numbers $\log(2), \log(3), ..., \log(n)$. What is the dimension of $V(100)$? of $V(n)$ for a general n?

Problem M3. Compute the limit

$$L = \lim_{n \to \infty} \left(\prod_{k=1}^{n-1} \cos \frac{\pi k}{2n} \right)^{1/n}.$$

Problem M4. Three points A, B, C are chosen at random (with uniform distribution) and independently inside a circle with center O.

(a) What is the probability q that O is inside the triangle ABC?

(b) What is the probability p that C is inside the triangle OAB?

(c) What is the probability r that the points O, A, B, C are vertices of a convex 4-gon?

Problem M5. (a) Find the number of matrices of size n by n over the field of two elements which are conjugate to a diagonal matrix. What is the answer for $n = 4$?

(b) What is the number of n by n matrices conjugate to a diagonal matrix over any finite field F_q?

Problem M6. At a banquet, a round table is served for an even number n of guests, and there are cards on the plates with guest names. However, the guests seat themselves without paying attention to the cards.

(a) Show that the table can be rotated so that at least two guests are seated according to the cards.

(b) Show that if n is any odd number, then statement (a) is false.