ON THE STRONGLY REGULAR GRAPH OF PARAMETERS $(99, 14, 1, 2)$

SUZY LOU AND MAX MURIN

Abstract. In an attempt to find a strongly regular graph of parameters $(99, 14, 1, 2)$ or to disprove its existence, we studied its possible substructure and constructions.

1. Introduction

Throughout the paper, the character $\sim$ will denote adjacency; G will denote the graph with the parameters under question, assuming it exists; and V will denote the vertex set of G.

Definition 1.1. A strongly regular graph with parameters (n, k, λ, μ) is a k-regular graph on n vertices such that a pair of vertices has λ neighbors in common if they are adjacent, and μ neighbors in common otherwise. There are many parameter sets for which it can be proven that no strongly regular graph exists, but for many other parameter sets, neither existence nor existence of a corresponding strongly regular graph has been shown. One of these parameter sets is $(99, 14, 1, 2)$.

Though these graphs are easy to define, it is not yet well understood for which parameter sets there exist at least one corresponding graph. Because of this lack of understanding, it is desirable to find out whether or not a parameter set such as $(99, 14, 1, 2)$ might correspond to a graph. In addition, it is unknown whether there is a Moore graph, a graph with diameter k and girth $2k+1$, with 57 vertices and girth 5. If this graph exists, it would be strongly regular and would complete the classification of Moore graphs. Though this particular problem is not related to Moore graphs, these two facts contribute to the interest in strongly regular graphs.

In studying this graph, we created several unsuccessful attempts at construction. We also found certain properties, such as bounds for the chromatic number and the size of a maximal independent set, possible substructures, and possible orders of automorphisms. In Section 2 we will first examine the ways the strongly regular graph of parameters $(9, 4, 1, 2)$ could potentially be a substructure. In Section 3 we will discuss attempted constructions with Fano-planes, followed by a discussion in Section 4 of maximal independent sets and a discussion of a triangle decomposition in Section 5. Section 6 will contain a discussion of possible orders of automorphisms in the graph. Section 7 will discuss the relationship of G with rotational block designs, and finally Section 8 will discuss the structures that arise from an automorphism of order 7.

2. The srg$(9, 4, 1, 2)$ as a substructure

Let H be the unique strongly regular graph of parameters $(9, 4, 1, 2)$.

Key words and phrases. Strongly regular graphs.
The project was supported by the PRIMES-USA program of MIT.

Theorem 2.1. *If G contains H minus an edge as a subgraph, then it contains H as an induced subgraph.*

Proof. Suppose, for the sake of contradiction, that there H minus an edge was an induced subgraph in G.

According a lemma of Wilbrink and Brouwer[1], the following equation holds for an induced subgraph of a strongly regular graph with parameters (n, k, λ, μ), such that the induced subgraph has N vertices, of degree $d_1, \ldots, d_N$, and M edges:

$$(n-N)-(kN-2M)+\lambda M+\mu\left(\binom{N}{2}-M\right)-\sum_{i=1}^{N}\binom{d_i}{2}=x_0+\sum_{j=3}^{N}\binom{j-1}{2}x_j,$$

where x_j denotes the number of vertices outside the subgraph adjacent to exactly j vertices in the induced subgraph.

One may verify that no vertex outside of the subgraph is adjacent to more than 3 vertices in this particular subgraph; otherwise, at least one of the parameters is violated. Therefore, applying the lemma above, $x_0 + x_3$=5.

The induced subgraph is illustrated in Fig. 1.

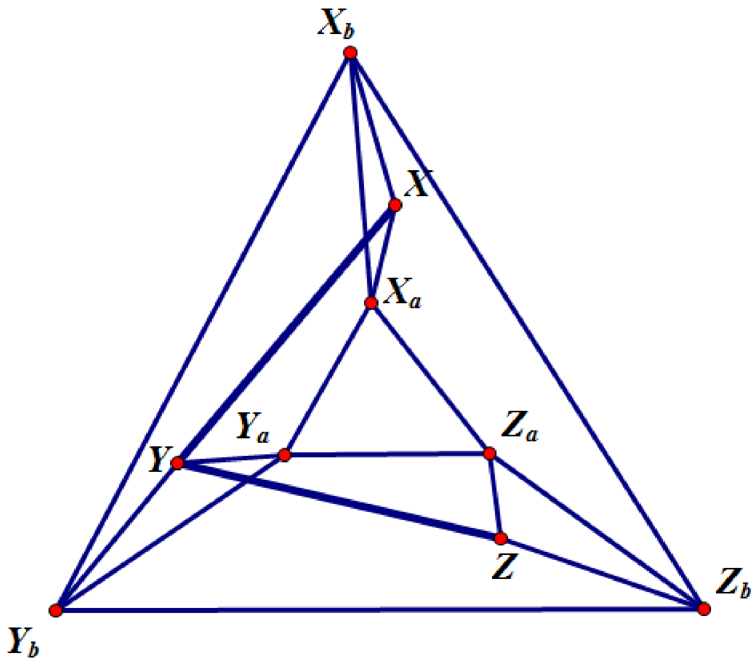

FIGURE 1. An illustration of the induced subgraph, which we shall prove does not exist in G.

For convenience, the induced subgraph on the vertices labeled X_a, X_b, Y_a, Y_b, Z_a, Z_b will be referred to as the "prism." Consider the bold edges. Each of these edges must form a triangle with another vertex. Keeping the third and fourth parameters in mind, we find that the two vertices that form a triangle with these edges are not adjacent to any vertex of the triangular prism and do not coincide.

Vertices X and Z share vertex Y as a common neighbor and have one more common neighbor. Again by examining the third and fourth parameters we find that this other common neighbor is not adjacent to any vertex of the triangular prism and does not coincide with the vertices previously mentioned.

This results in the subgraph shown in Fig. 2:

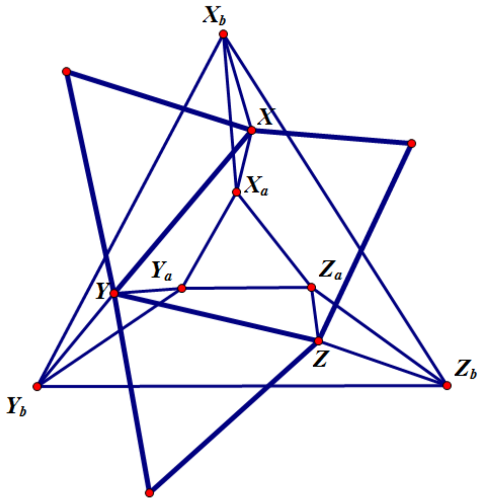

FIGURE 2. A more detailed subgraph that must exist in G if the induced subgraph shown in Fig. 1 exists.

Let S be the set of the 60 vertices such that they are the neighbors of a vertex in the prism, that are themselves not in the prism.

By the fourth parameter, X and Z are each adjacent to 2 vertices in S. Similarly, vertex Y is adjacent to no such vertices. We find that X and Z are hence adjacent to 9 vertices that are not in S, and for both X and Z, 2 of these 9 neighbors are already drawn in the diagram. Similarly, Y has 10 neighbors that do not belong to S, and two of them are drawn in the diagram.

Given this information, we can directly compute x_0: it is 5. Thus, $x_3 = 0$.

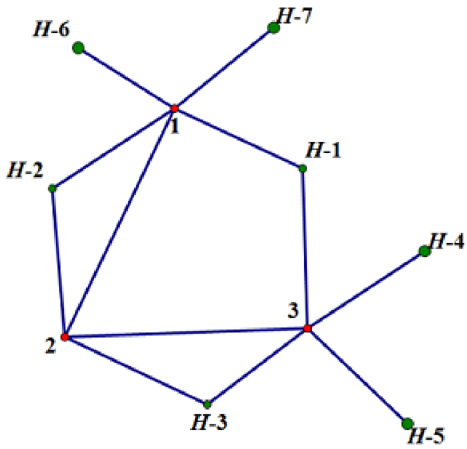

FIGURE 3. All the vertices that have 2 or more neighbors in $S \cap \{1, 2, 3\}$

Consider a vertex v belonging to this set of five vertices. By the fourth parameter, exactly twelve of its neighbors belong to S. We also know that v shares two neighbors with each of X, Y, Z. How is this possible? That would seem to make 18 neighbors of v, so we must be overcounting. At least 4 neighbors of v must either be adjacent to two of X, Y, Z, or simultaneously be adjacent to one of X, Y, and Z and belong to S. (Recall it is impossible for a vertex belonging to S to be adjacent to more than one of X, Y, and Z.)

That is to say, each of these five are adjacent to four of the green vertices in the subgraph shown in Fig. 3, where $H-4$, $H-5$, $H-6$, $H-7$ are the vertices that simultaneously are adjacent to one of X, Y, and Z and belong to S.

Choosing 4 vertices from 7 is the same as not choosing 3 from the 7, and we note that because of the fourth parameter, at least 2 members of the set $\{H-1, H-2, H-6, H-7\}$ must not be chosen, and similarly, at least two members of the set $\{H-1, H-3, H-4, H-5\}$ must not be chosen. That means that $H-1$ can never be chosen. Then $H-1$ has no neighbors from the vertices constituting x_0. However, since $H-1$ does not belong to S, we can count its neighbors: 2 are shown in the previous diagram; by the fourth parameter, 8 of its neighbors belong to S. Its remaining neighbors must be adjacent to one or more of X, Y, and Z, but not to any vertices of the triangular prism. But it already shares two neighbors with Y, and for each of vertex X and vertex Z it needs only 1 more common neighbor. Then $H-1$ has only 12 neighbors, contradiction.

□

3. Labelings with Fano planes

Definition 3.1. A Fano plane is a set of seven 3-element subsets, called lines, of $\{1, \ldots, 7\}$ such that every pair of lines share exactly one element. The elements are also called points.

There are 30 distinct Fano planes, which can be grouped into two disjoint sets of 15 Fano planes, such that two Fano-planes in the same set have the following property: the two Fano planes share exactly one 3-element set, one Fano plane can be obtained from the other by cyclically permuting the three elements of this shared 3-element set. In addition, if we cyclically permute three elements of a single set of a Fano plane, the result is a Fano plane in the same set of 15 Fano planes as the initial Fano plane.

Suppose one of these disjoint sets of 15 Fano planes is $\{F_0, F_1, \ldots, F_{14}\}$. Suppose that F_0, F_{2n-1}, and F_{2n} share a line for $n \in 1, \ldots, 7$. Then we can label G as follows: Call a central vertex F_0, and its 14 neighbors $F_0 \ldots F_{14}$, such that F_0, F_{2n-1}, and F_{2n} form a triangle.

For the other 84 vertices in the graph, suppose a vertex is the common neighbor of F_i and F_j. If F_k is the Fano-plane such that F_i, F_j, F_k all share a line, and e is the shared line, then label the vertex as (F_k, e).

One attempt at construction was to create rules for adjacency among the vertices (F_k, e). However, none of the rules that were tried worked. A few rules that were the most noteworthy were the following.

1. Consider neighbors of F_x of the form (F, l). Connect two of them if the line the Fano-plane portion of their labellings share is in F_0. This rule is equivalent to the impossible construction with $\mathrm{srg}(9, 4, 1, 2)$.

2. Connect (F_x, l_m) and (F_y, l_n) if l_m and l_n are disjoint.

3. Consider F_x and F_y that form a triangle with F_0. Connect a neighbor of F_x of form (F, l) and a neighbor of F_y of form (F, l) if the Fano-plane part of their label is the same.

4. The following is not a rule, but rather a set of conditions for a rule.

Let e be the line F_k shares with F_0. In that case, F_k can be obtained from F_0 by cyclically permuting e. Consider the vertex $e \cap l$, which we will call P. Define the root of a vertex (F_k, l) to be the point such that P occupies the position it previously occupied before the rotation.

The root has the following property: Consider any one of the four lines of F_k that do not contain P, and consider two points of that line, a and b. Then a, b, and the root do not form a line in F_0. It is the only point with this property.

Consider (F_k, l); suppose its root is P and that it is connected to F_a and F_b. Then consider all the lines that contain P but are not included in F_0. There are 12 such lines. For each line l_m that is one of the twelve, consider the three vertices such that l_m is included in their label. Choose one of them whose root is part of l and connect (F_k, l) to it.

The reason these conditions cannot be met is the condition that (F_k, l) shares exactly one vertex with F_a and with F_b: there is no way to meet this condition.

4. Independent sets and 2-block designs of parameters $(22, 4, 2)$.

Theorem 4.1. *An independent set in G of size 9 cannot be maximal.*

Proof. Suppose there were such a maximal independent set, I. Let x_i be the number of vertices in $V \setminus I$ adjacent to exactly i vertices in I, and let y_i denote the set of vertices with i neighbors in I. For $i \geq 6$, $x_i = 0$. This is because a vertex v_j with j neighbors in I needs $2(9-j)+j = 18-j$ common neighbors with the vertices of I, and has $14-j$ neighbors in $V \setminus I$. But if v_j had a neighbor with 6 or more neighbors in I, then it would have to have at least $19-j$ common neighbors with the vertices of I. Thus, if there were a vertex with 6 or more neighbors in I, it would have no neighbors in $V \setminus I$, which is clearly impossible.

Lemma 4.2. *Related to the above observation, for all vertices v_j in $V \setminus I$, if a, b, c, and d are the number of neighbors of v_j with, respectively 2, 3, 4, and 5 neighbors in I, $a + 2b + 3c + 4d = 4$. This also means that a vertex in $V \setminus I$ can never serve as a common neighbor for a vertex in y_5 and a vertex with 2 or more neighbors in I.*

Proof. A vertex v_j with j neighbors in I needs $2(9-j)+j = 18-j$ common neighbors with the vertices of I. The actual number of common neighbors is $2a+3b+4c+5d+(14-j-a-b-c-d)$. Setting this equal to $18-j$ yields the above. □

We have

$$x_1 + x_2 + x_3 + x_4 + x_5 = 90$$
$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 9 * 14 = 126$$
$$x_2 + 3x_3 + 6x_4 + 10x_5 = 2\binom{9}{2} = 72.$$

In addition, we have $x_5 \leq 3$. Suppose, on the contrary, we had 4 vertices A, B, C, D such that each had 5 neighbors in I. If A and B shared only one common neighbor, no vertex could have more than 4 neighbors in I without sharing 3 neighbors with A or B. Thus, they must share 2 common neighbors. Then C and D must both be adjacent to the vertex of I adjacent to neither A nor B, as well as 2 vertices adjacent to A only and 2 vertices adjacent to B only. Then C and D share 3 common neighbors, contradiction. Note that if $x_5 = 3$, then any pair of vertices with 5 neighbors in I

share two common neighbors in I. In this case, 6 vertices in I have 2 neighbors in y_5, and 3 vertices in I have 1 neighbor in y_5.

The only solutions to the above equations are: $(78,0,0,12,0)$, $(77,0,6,4,3)$, $(77,2,0,10,1)$, $(76,4,0,8,2)$, $(77,1,3,7,2)$, $(79,3,3,5,3)$, $(75,6,0,6,3)$.

However, none of these solutions work. In the first solution, $(78,0,0,12,0)$, consider the 12 vertices with 4 neighbors in I. For every time a vertex in I is adjacent to such a vertex, it gains a common neighbor with a different vertex in I 3 times. In total, it must gain $2*8=16$ common neighbors in I, but 16 is not divisible by 3, so this is impossible.

Now note that the number of edges from $y_2 \cup y_3$ to $y_4 \cup y_5$ is at most $|y_2 \cup y_3|$, because each vertex of $y_2 \cup y_3$ has at most one neighbor in $y_4 \cup y_5$. On the other hand, it is also at least $|y_4|-|y_5|$, because at most $|y_5|$ members of y_4 have a neighbor in y_5, and the rest must have a neighbor in $y_2 \cup y_3$. For the third, fourth, and fifth solutions, this causes an immediate contradiction.

In the second solution, $(77,0,6,4,3)$, Lemma 4.2 implies that the 3 vertices in y_5 each have 2 neighbors in y_3, and none in y_4. Thus every vertex in $y_4 \cup y_5$ must have a neighbor in $y_2 \cup y_3$, but this means there are at least 7 edges between $y_2 \cup y_3$ and $y_4 \cup y_5$, while on the other hand there are at most 6; contradiction.

In the sixth solution, $(79,3,3,5,3)$, consider the 5 vertices in y_4. As mentioned before, 6 vertices in I have 2 neighbors in y_5, and 3 vertices in I have 1 neighbor in y_5. Each vertex in y_4 must have at least 2 neighbors in I that have 1 neighbor in y_5, or else it shares at least 3 neighbors with a vertex in y_5. Thus, each serves as a common neighbor either 1 or 3 times for the 3 vertices in I that have 1 neighbor in y_5. These 3 vertices share a common neighbor a total of 6 times, so we see that each of the 5 vertices in y_4 has exactly 2 neighbors out of these 3 vertices in I. Then none of them is adjacent to a vertex in y_5. Then the only way for Lemma 4.2 to be fulfilled with respect to the vertices in y_5 is for a vertex to be adjacent to 2 vertices in y_3, or 1 vertex in y_3 and 2 vertices in y_2. Then 2 vertices in y_5 share a common neighbor in y_3 or y_2, as well as 2 common neighbors in I, contradiction.

In the seventh solution, $(75,6,0,6,3)$, as above, the 6 vertices in y_4 must each be adjacent to exactly 2 vertices in I that have 1 neighbor in y_5. Thus, no edges exist between y_4 and y_5. Then to fulfill the lemma, each vertex in y_5 has 4 neighbors in y_2. Then 2 vertices in y_5 share at least 2 neighbors in y_2 as well as 2 neighbors in I, contradiction.

Thus, all possibilities lead to a contradiction. □

Theorem 4.3. *The largest independent set of a strongly regular graph of parameters* $(99,14,1,2)$ *has size at most* 22. *If it has size* 22 *then every vertex not belonging to the independent set has exactly* 4 *neighbors in the independent set.*

Proof. Let I be a maximal independent set with n vertices. The set S of all vertices with at least one neighbor in I has size $99-n$. The number of edges between S and I is $14n$.

Suppose $S=\{s_1,\ldots,s_{99-n}\}$. Let $F(i)$ be the number of neighbors s_i has in I. Therefore, $\sum_{i=1}^{99-n} F(i) = 14n$. In addition, because 2 nonadjacent vertices share 2 neighbors, $\sum_{i=1}^{99-n} \binom{F(i)}{2} = 2\binom{n}{2} = n^2-n$. Therefore, $\sum_{i=1}^{99-n} F(i)^2 = 2n^2+12n$.

By the RMS-AM inequality, $\sqrt{\frac{\sum_{i=1}^{99-n} F(i)^2}{99-n}} = \sqrt{\frac{2n^2+12n}{99-n}} \geq \frac{\sum_{i=1}^{99-n} F(i)}{99-n} = \frac{14n}{99-n}$. After some algebra, this turns into $-n^2-5n+594 \geq 0$, so $-27 \leq n \leq 22$. Thus, the maximum size of I is 22, as desired. Equality holds when $F(i)$ is equal across all values of i; thus, for all i, $F(i) = \frac{14*22}{77} = 4$. □

The 77 vertices outside the independent set all have 4 neighbors in S.

5. Miscellaneous.

Assuming existence, G has a unique triangle decomposition. Consider a triangle and all triangles adjacent to it. This is illustrated in Fig. 4.

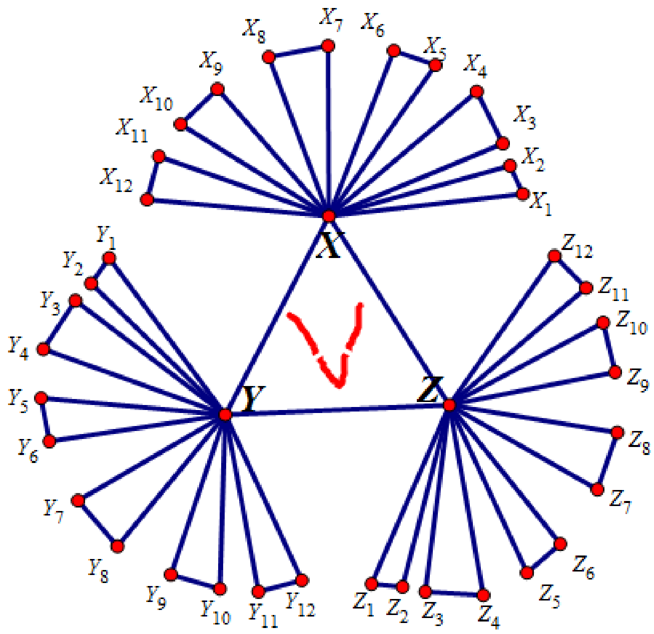

Figure 4. An illustration of a triangle in G and all adjacent triangles.

Since there are $\frac{99*14}{2} = 693$ edges in the graph, there are $\frac{693}{3} = 231$ disjoint triangles in the graph. We can consider a graph T on 231 vertices such that each triangle in G is a vertex of T and two vertices in T are adjacent iff the corresponding triangles in G share a vertex. The graph T is 18-regular, and because G has diameter 2, T has diameter 3. In the above diagram, a central vertex in T is called v.

Lemma 5.1. *The chromatic number of G is between 5 and 11.*

Proof. The lower bound of 5 is a direct consequence of the fact that the maximum size of an independent set in G is 22.

By the second parameter, there is a perfect matching between the vertices X_i and Y_i. Similarly, there is a perfect matching between the vertices Y_i and Z_i, and X_i and Z_i. Then these edges determine disjoint cycles of total length 36, each cycle of length divisible by 3.

As a result, the induced subgraph on the 36 vertices is 3-regular. By Brook's Theorem, these vertices can be 3-colored. Assign them an arbitrary 3-coloring.

Let the set $V' = V \setminus \{X_1, \ldots, X_{12}, Y_1, \ldots, Y_{12}, Z_1, \ldots, Z_{12}\}$. Now, consider the vertices of V'. By the fourth parameter, each one has two neighbors of the form X_i, two neighbors of the form Y_i, and two neighbors of the form Z_i. Thus, the induced subgraph on these vertices is 8-regular. Again by Brook's Theorem, these vertices can be 8-colored. Assign them an arbitrary 8-coloring with colors that have not been used before.

Now consider X, Y, and Z, which have not been assigned colors. Assign them 3 distinct colors from the 8-coloring on the vertices in V'.

Thus, the graph is 11-colorable. □

Let us shift our attention again to T, so that a vertex refers to a triangle, and a G-vertex refers to a vertex in the traditional sense. Consider the set of vertices distance 2 from v. Now define the following:

α : The number of such vertices connected to exactly one G-vertex distance 1 from v.

$\alpha - vertex$: A vertex with the above property.

β : The number of such vertices connected to exactly two G-vertices distance 1 from v.

$\beta - vertex$: A vertex with the above property.

γ : The number of such vertices connected to exactly three G-vertices distance 1 from v.

$\gamma - vertex$: A vertex with the above property.

One may easily verify the following equations:

$$\alpha + \beta = 180$$
$$\beta + 3\gamma = 36$$
$$\alpha - 3\gamma = 144.$$

Also, the number of vertices distance 3 from v is $32 - \gamma = 20 + \frac{\beta}{3}$. The value γ is an integer between 0 and 12. Note that γ cannot be 11: as we noted before, the perfect matchings between the vertices of form X_i, Y_i; Y_i, Z_i, and X_i, Z_i fall into disjoint cycles of lengths divisible by 3 and summing to 36. But if $\gamma = 11$, we have 11 3-cycles and three loose edges that do not fall into cycles; contradiction.

If $\gamma = 12$ for all vertices in T, then this reduces to the impossible labeling with srg(9, 4, 1, 2). This is apparent after relabeling the previous diagram as in Fig. 5.

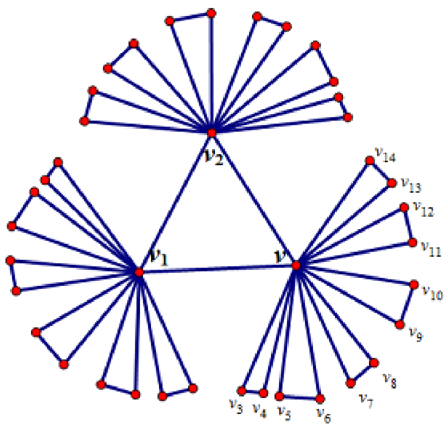

FIGURE 5. A relabeling of the diagram in Fig. 4 that demonstrates the connection between $\gamma = 12$ and the first structure discussed.

6. Possible orders of automorphisms of G

Theorem 6.1. $\mathrm{srg}(99, 14, 1, 2)$ *has no automorphisms of* $p > 14$*, where* p *is prime.*

Proof. An automorphism of the graph G of order p must have at least one orbit of order p, since p is prime. However, not every point of the graph can be in such an orbit, since $p \nmid 99$. Since the graph is connected, at least one point in an orbit must connect to a point P not in an orbit. However, by applying the automorphism, we can see that the P connects to every point in the orbit, so $\deg P = 14 \geq p > 14$. This is a contradiction, so no automorphisms of order p exist. □

Theorem 6.2. $\mathrm{srg}(99, 14, 1, 2)$ *has no automorphisms of order* 13.

Proof. Assume such an automorphism π exists. Then, as before, we must have at least one automorphism of order 13, and points not in any orbits. By connectedness, there must exist a point P which connects to an orbit; thus, P must connect to every point in that orbit. P must also connect to exactly one more point, which cannot be in an orbit. By the property $\lambda = 1$, P must share a common neighbor with each of these points. Thus, two points A and B in the orbit must be connected to each other. Since π has order 13, and A and B are in the same orbit, there exists a positive integer $n < 13$ such that $\pi^n(A) = B$. Since π is an automorphism, π^n is also an automorphism. Thus, B connects to $\pi^n(B)$. However, P and B are connected, and have two common neighbors: A and $\pi^n(B)$. This is a contradiction, so π cannot exist. □

Theorem 6.3. $\mathrm{srg}(99, 14, 1, 2)$ *has no automorphism of order* 11.

Proof. Assume that some such automorphism π exists. Let n be the number of orbits of size 11 of π. Then, $1 \leq n \leq 9$. First, examine the case that $n < 9$. In this case, orbits of size 1 exist. By connectedness, there must be a point P that connects to an orbit. Each of the points in the orbit must connect to another neighbor of P. Since P has 3 neighbors outside of the orbit and 11 inside it, there must be two points in the orbit that connect. As before, contradiction. Thus, $n = 9$.

Let us label the orbits A_1 through A_9. Then, let us define a matrix M by M_{ij} being the number of points in A_i that any point in A_j connects to. Note that $M_{ij} = M_{ji}$, so M is symmetric.

If any two points in the same orbit A_i, P and Q, are connected, then there exists a j such that $P = \pi^j(Q)$. Then, since π is an automorphism, $\pi^j(P) \sim \pi^j(Q) = P$. $\pi^j(P)$ is also in A_i. If the same process is repeated with P and $\pi^j(P)$, we get back Q. Therefore, for any point in A_i, all of its neighbors in A_i can be paired, and thus M_{ii} is even for all i.

Let us consider the eigenvalues of M. The eigenvalues of the adjacency matrix of G are 14, 3, and -4. Any eigenvector of M corresponds to an eigenvector of G, the correspondence being to set every point in A_i to the corresponding value in the eigenvector of M. By the definition of M, the eigenvalue must also be equal: therefore, the eigenvalues of M must be in the set $\{14, 3, -4\}$. Since the adjacency matrix of G has the eigenvalue 14 with multiplicity 1, M can have this eigenvalue with multiplicity at most 1; the vector $\langle 1, 1, 1, 1, 1, 1, 1, 1, 1\rangle$ has this eigenvalue, so the multiplicity of the eigenvalue 14 is exactly 1. Next, note that the sum of the eigenvalues is equal to the trace, and that the diagonal of M must contain only positive even integers. Thus, the sum of the eigenvalues must be an even positive integer that is a sum of 14 and eight values from the set $\{3, -4\}$. The only such even positive sums are 38, 24, and 10.

Let us now consider the matrix M^2. Since M_{ij} counts the number of ways to get from one specific point in A_i to any point in A_j by a path of length exactly 1, $(M^2)_{ij}$ counts the number of ways to get from any specific point in A_i to some point in A_j along a path of length exactly 2. In other words, $(M^2)_{ij}$ is the number of common neighbors one specific point of A_i has with all of the points of A_j. First, consider the case that $i = j$. Any point has degree 14, so it has 14 common neighbors with itself. Then, any point of the orbit A_i is connected to M_{ii} points on the same orbit by the definition of M. For each one it is connected to, it has 1 common neighbor; otherwise, it has 2. Thus, $(M^2)_{ii} = 14 + M_{ii} + 2(10 - M_{ii}) = 34 - M_{ii}$. If $i \neq j$, then $(M^2)_{ij} = M_{ij} + 2(11 - M_{ij}) = 22 - M_{ij}$, similarly. Therefore, $(M^2 + M)_{ij} = 34$ if $i = j$ and 22 otherwise.

By definition,

$$(M^2)_{ii} = \sum_{j=1}^{9} M_{ij}^2 = 34 - M_{ii}.$$

This implies that every value in M must be at most 5.

Since the degree of any vertex is 14, the sum of the values of any row must be equal to 14. By trying every possibility, it can be shown that there are only seven possible rows of M that fulfill these two equations, up to permutation:

$$\begin{array}{c} 0,4,3,2,1,1,1,1,1 \\ 0,4,2,2,2,2,1,1,0 \\ 0,3,3,3,2,1,1,1,0 \\ 0,3,3,2,2,2,2,0,0 \\ 2,4,2,2,1,1,1,1,0 \\ 2,3,3,2,2,1,1,0,0 \\ 4,2,2,1,1,1,1,1,1 \end{array}$$

The first value in each listing must lie on the diagonal.

Note that the value 5 occurs in no row. Therefore, it is never possible to have a value of 5 in M. Thus, the sum of the eigenvalues can never be 38, so it must be either 24 or 10.

Since we know every possible row, we can now try every possibility. No such matrix exists. Therefore, there is no automorphism of order 11. □

7. Block designs of the parameters $(22, 4, 2)$.

Definition 7.1. A block design of parameters $(22, 4, 2)$ is comprised of a set S of 22 values (for convenience let them be the integers from 0 to 21), called treatments, and a set B of 77 4-subsets of S, called blocks, such that every value k in S is in exactly 14 members of B; and every pair of distinct values in S is in exactly 2 members of B.

Let G be some srg$(99, 14, 1, 2)$ that has an independent set S of size 22. Then, let $G \backslash S$ be B. Let the graph G' be G with every edge between two points of B removed; G' is bipartite with parts S and B. As noted earlier, every vertex in B must have degree 4 in G'. Let $B' = \{\{k \in S \mid k \sim b\} \mid b \in B\}$. Since the srg parameter μ is 2, every pair of members of S must have two common neighbors. Thus, (S, B') is a

$(22, 4, 2)$-block design. Note, however, that not every block design corresponds with a potential G': some block designs have repeated blocks, or blocks that share three elements: this would lead to two points in B having four or three common neighbors, respectively.

Every possible graph that has an independent set of size 22 can thus be associated with a block design. One notable family of block designs is the family of cyclic, or 1-rotational, block designs, which have the property that if any block $\{a_1, a_2, a_3, a_4\}$ is in the design, then the block $\{a_1 + 1, a_2 + 1, a_3 + 1, a_4 + 1\}$ is also in the design (addition modulo 22). Some members of this family have repeated blocks or blocks that share three members; these do not form viable graphs.

Every block in a cyclic block design is part of a family of blocks produced by adding one to each element. For a block b, let $f(b)$ be the block generated by adding 1 to every element of b modulo 22. Clearly, $f^{22}(b) = b$. Thus, the order of b under f must divide 22. If the order was 1, then for some k in b, $k + 1$, $k + 2$, $k + 3$, and $k + 4$ would also have to be in b, contradiction. A similar contradiction arises for an order of 2. Thus, the orders of all blocks must be either 11 or 22. If the order is 11, then the block must be equal to $\{k_1, k_1 + 11, k_2, k_2 + 11\}$; if it is not of this form, it must have order 22.

Another family of block designs is the family of 2-rotational block designs, with the property that if any block $\{a_1, a_2, a_3, a_4\}$ is in the design, then the block $\{a_1 + 2, a_2 + 2, a_3 + 2, a_4 + 2\}$ is also in the design. Thus, the family of 2-rotational block designs is a superset of the family of cyclic block designs. Defining the function $g(b)$ as $f(f(b))$, with f as before, then every block in such a design must have $g^{11}(b) = 1$. As before, no block can have an order of 1, so every block in such a block design has order 11.

An attractive potential construction of G from a 2-rotational block design is to require that if two points in G' labeled by blocks b_1 and b_2 are connected, then so are $g(b_1)$ and $g(b_2)$. This, however, would mean that g would be an automorphism of G with order 11. As shown above, this is impossible, so the most attractive construction does not work. It might still be possible to create an srg$(99, 14, 1, 2)$ from a 2-rotational block design through some method.

8. Automorphisms of order 7.

Assume that some automorphism π of G of order 7 existed. Then, since 7 does not divide 99, there must be at least one orbit of size 1. Since the graph is connected, at least one orbit of size 1 must connect to an orbit of size 7. Let us call the point in this orbit P, and let us call the orbit of size 7 to which P connects A. Since P is connected to a point in A, it must have exactly one common neighbor with that point. No element of A can connect to any other orbits of size 1, because then P would have seven common neighbors with that orbit. Thus, the common neighbor must be in an orbit of size 7; call that orbit B. It is connected to A and P.

Therefore, P connects to two orbits of size 7, A and B; P has degree 14, so it does not connect to any other orbits of size 1. Thus, any other orbits of size 1 must be connected to two orbits of size 7 similarly. Thus, if there are k orbits of size 1, there must be at least $2k$ orbits of size 7. This requires $15k$ points, so $k \leq 99/15 < 7$. The number of orbits of size 7 is $(99 - k)/7$, which must be an integer, so k must be 1 modulo 7. Therefore, k must equal 1, and thus P is the only orbit of size 1.

Let us label the members of A and B as A_1 through A_7 and B_1 through B_7 respectively, so that $\pi(A_i) = A_{i+1}$. We know that every point in A must connect to a point in B, so WLOG let A_i connect to B_i. No A_i can also connect to any B_j for $i \neq j$, since that would cause A_i and P to have two common neighbors, even though they are connected to each other. Similarly, A_i is not connected to A_j for any j.

Since A_i and B_j are not connected for $i \neq j$, they must share two common neighbors, one of which must be P. Let this common neighbor be called $Q_{i,j}$. Note that A_{i+1} and B_{j+1} have the common neighbor $Q_{i+1,j+1}$, but also $A_{i+1} = \pi(A_i)$ and $B_{j+1} = \pi(B_j)$, so $Q_{i+1,j+1} = \pi(Q_{i,j})$.

Let us define C^α_β as $Q_{-\alpha+\beta,\alpha+\beta}$ for $\alpha \in \{1, 2, 3\}$. Then, $C^\alpha_{\beta+k} = Q_{-\alpha+\beta+k,\alpha+\beta+k} = \pi^k(Q_{-\alpha+\beta,\alpha+\beta}) = \pi^k(C^\alpha_\beta)$. Also, $A_i \sim C^\alpha_{i+\alpha}$ and $C^\alpha_i \sim B_{i+\alpha}$. Define D^α_β as $Q_{\alpha+\beta,-\alpha+\beta}$ for $\alpha \in \{1, 2, 3\}$. As before, $D^\alpha_{\beta+k} = \pi^k(D^\alpha_\beta)$. Also, $B_i \sim C^\alpha_{i+\alpha}$ and $C^\alpha_i \sim A_{i+\alpha}$.

Now, note that any A_i and A_j for unequal i and j must share a common neighbor; let this be $R_{i,j} = R_{j,i}$. Let us similarly define the neighbor of B_i and B_j as $R'_{i,j}$. As before, $R_{i+k,j+k} = \pi^k(R_{i,j})$. Then, we can define E^α_β as $R_{-\alpha+\beta,\alpha+\beta}$ for $\alpha \in \{1, 2, 3\}$. Therefore, as before, $E^\alpha_(\beta + k) = \pi^k(E^\alpha_\beta)$. Similarly define F^α_β as $R'_{-\alpha+\beta,\alpha+\beta}$. Once again, $F^\alpha_(\beta + k) = \pi^k(F^\alpha_\beta)$.

Let us now define a 15 by 15 matrix M. The columns and rows correspond to P, A, B, C^1, C^2, C^3, D^1, D^2, D^3, E^1, E^2, E^3, F^1, F^2, and F^3 in that order, and M_{ij} is defined as the number of times any point in the ith orbit is adjacent to some point in the jth orbit.

Lemma 8.1. *Either* $M_{ii} = 0$ *or* $M_{ii} = 2$.

Proof. If there is an edge within C_i, then clearly there is a 7-cycle within C_i. If there are 2 7-cycles, then it is easy to verify that the parameters of the graph are violated. □

Further note that no point in P, A, or B is adjacent to any other point in its own orbit, so $M_{11} = M_{22} = M_{33} = 0$. Thus, the trace is at most $12 \times 2 = 24$, and as before, must be even. The eigenvalues of M must be a subset of the eigenvalues of G, as shown in the proof of Theorem 6.3, so M has one eigenvalue equal to 14, the rest being either 3 or -4. If a of the eigenvalues are -4, then the trace is equal to $14 + 42 - 7a$, which must be divisible by 7. Thus, the trace is either 0 or 14.

9. Acknowledgements

We are grateful to MIT PRIMES-USA for providing us with this project, and to Dr. Peter Csikvari for suggesting the problem and mentoring us.

References

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