Belyi functions with prescribed monodromy

Ravi Jagadeesan,
Mentor: Akhil Mathew

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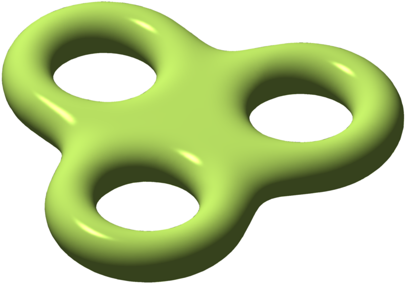
Compact Riemann surfaces

Definition

A **Riemann surface** is a one-dimensional complex manifold.



[1]

[1] http://upload.wikimedia.org/wikipedia/commons/f/f0/Triple_torus_illustration.png.

Algebraic curves

- Loosely speaking, an **algebraic curve** is a one-dimensional object that is the set of common zeros of a finite set of multivariate polynomials.

$$y^2 = x^3 - x$$

Theorem (Riemann Existence)

Every compact Riemann surface has an algebraic structure.

- An algebraic curve is **defined over** $\overline{\mathbb{Q}}$ if the equations can be taken to have coefficients in $\overline{\mathbb{Q}}$.

Belyi functions

Definition

A **Belyi function** is a morphism $f : X \to \mathbb{P}^1_{\mathbb{C}}$,

- X a compact Riemann surface (smooth, projective, irreducible curve over $\mathbb{C}$)
- f unbranched outside $\{0, 1, \infty\}$.

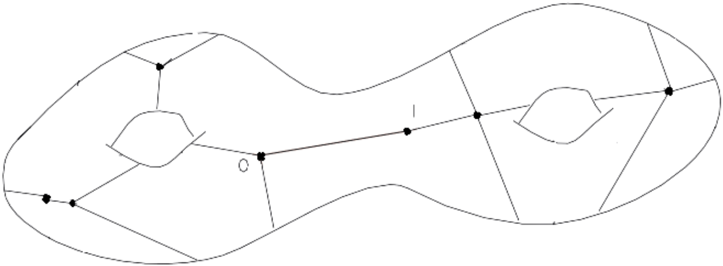
Theorem (Belyi)

An algebraic curve over $\mathbb{C}$ admits a Belyi function if and only if it is defined over $\overline{\mathbb{Q}}$.

Dessins d'enfants

Definition

A **dessin d'enfant** is a connected, bipartite graph G embedded as a map into a topological compact oriented surface.



[2]

[2] M. M. Wood, *Belyi-Extending Maps and the Galois Action on Dessins d'Enfants*, Publ. RIMS, Kyoto Univ., **42**: 721737, 2006.

Correspondence between Belyi functions and dessins

Theorem (Grothendieck)

There is a natural one-to-one correspondence between isomorphism classes of Belyi functions and isomorphism classes of dessins d'enfants.

- Associate $f : X \to \mathbb{P}^1_{\mathbb{C}}$ to the graph $f^{-1}([0,1])$.
- Bipartite with parts $V_0 = f^{-1}(\{0\})$ and $V_1 = f^{-1}(\{1\})$

The full dictionary of objects

The following objects define equivalent data:

- isomorphism classes of dessin d'enfants with n edges
- isomorphism classes of Belyi functions of degree n
- conjugacy classes of transitive representations $\langle x, y \rangle \to S_n$
- conjugacy classes of subgroups of $\langle x, y \rangle$ of index n

Action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

- There is a natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the category of algebraic curves over $\overline{\mathbb{Q}}$.
  - Apply an automorphism of $\overline{\mathbb{Q}}$ to all the coefficients of the defining polynomials
  - Equivalently, base-change by an automorphism of $\operatorname{Spec} \overline{\mathbb{Q}}$
- By Belyi's Theorem, this gives an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the category of Belyi functions.
  - The action is faithful.
- Hence, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of isomorphism classes of dessins as well.
  - So, a number-theoretic object acts on a purely combinatorial object.

Application to inverse Galois theory

Question (Inverse Galois Problem)

What are all finite quotients of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*? What is* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*?*

One can study $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by its (faithful) action on the category of dessins.

Question (Grothendieck)

How does $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *act on the category of Belyi functions (set of isomorphism classes of dessins)?*

Galois invariants of dessins

Question (Grothendieck)

When are two dessins in the same Galois orbit?

To answer this, it suffices to find a perfect **Galois invariant**.

Three particularly simple Galois invariants are:

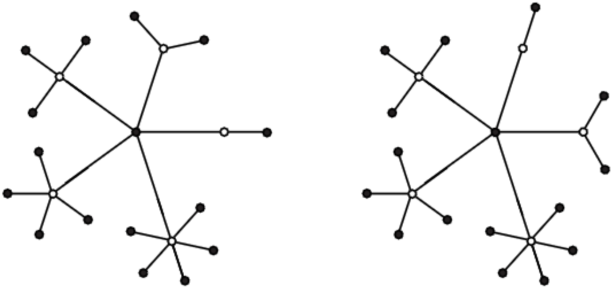
- degree multisets of V_0 and V_1, and the number of edges that bound each face of the dessin
  - equivalently, the cycle types of the monodromy generators
  - very simple to compute!
- monodromy group of a Belyi function
  - not as simple to compute
- rational Nielsen class

Precision of known Galois invariants

Question

How precise is the monodromy cycle type as a Galois invariant? What about other known invariants?

Precision of known Galois invariants



[3]

- not Galois conjugate, but share the same degrees, monodromy groups, and rational Nielsen classes
- distinguished by a different Galois invariant (due to Zapponi)

[3] L. Zapponi, *Fleurs, arbres et cellules: un invariant galoisien pour une famille d'arbres*, Compositio Mathematica **122**: 113-133, 2000.

The Main Theorem

Definition

For all positive integers N, let

$$\mathrm{Cl}(N) = \max_{n \le N} \max_{\lambda_1, \lambda_2, \lambda_3 \vdash n} \left(\begin{array}{c} \text{number of Galois orbits of} \\ \text{Belyi functions with monodromy} \\ \text{of cycle type } (\lambda_1, \lambda_2, \lambda_3). \end{array} \right)$$

Our Theorem

For all positive integers N, we have

$$\mathrm{Cl}(N) \ge \frac{1}{16} 2^{\sqrt{\frac{2N}{3}}}.$$

Future Directions

- Upper bound on $\mathrm{Cl}(N)$
- Consider Cartesian commutative squares of the form

$$\begin{array}{ccc} Y & \longleftarrow & X \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xleftarrow[f=\frac{(z+1)^2}{4z}]{} & \mathbb{P}^1 \end{array}.$$

  - For a fixed right morphism, consider all possible left morphisms
  - Leads to an extrinsic Galois invariant for the right morphism (used in the proof of the Main Theorem)
  - How does one define this invariant intrinsically?

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