Minimum Degrees of Minimal Ramsey Graphs

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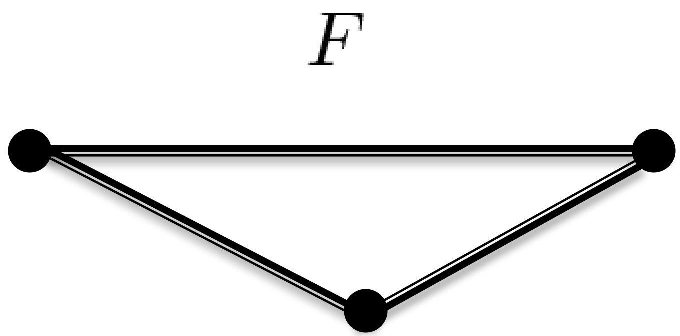
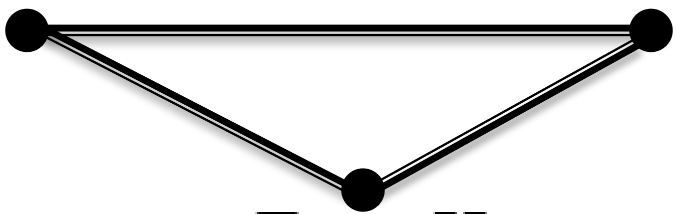
Introductory Information

- A graph F is a set of vertices V with an edge set $E \subseteq \binom{V}{2}$.

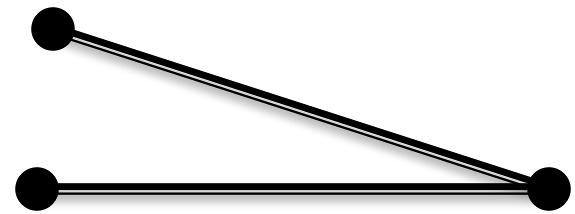


Introductory Information

F'



H



- Write $F \rightarrow H$ to mean that for any two coloring of the edges of F, there is a monochromatic copy of H.
- Removing any edge or vertex of F would make a new graph $F' \not\rightarrow H$.
- A graph which satisfies both of the above properties is said to be a minimal Ramsey graph for H.
- The family of all minimal Ramsey graphs for H is denoted by $M(H)$

Some Classical Problems

- Ramsey's Theorem: For any H , the set $M(H)$ is nonempty.
- Is $M(H)$ finite or infinite?
- Smallest number of edges contained in any graph in $M(H)$.
- The Ramsey number $R(H)$: the smallest number of vertices contained in any graph in $M(H)$.
  - One of the most important, central, and famous problems in combinatorics.

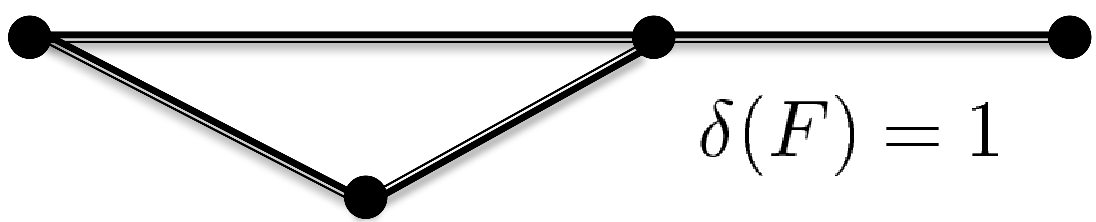
Introductory Information

- In this presentation, we are concerned with the value

$$s(H) := \min_{F \in M(H)} \delta(F)$$

- $\delta(F)$ is the minimum degree of the graph F .

F

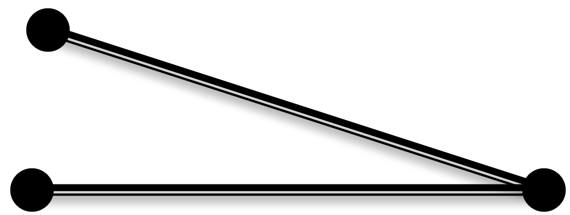


$\delta(F) = 1$

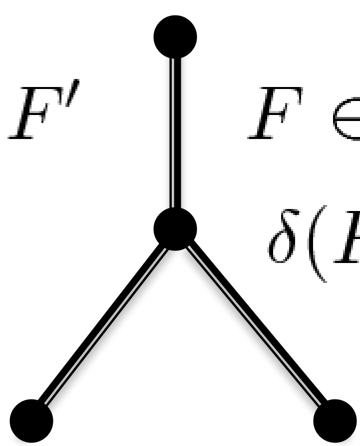
- In general, this has only been solved for a few classes of graphs.

Example of Finding $s(H)$

H



- Clearly, we must have $s(H) > 0$.
- Can we have $s(H) = 1$?



F'

$F \in M(H)$

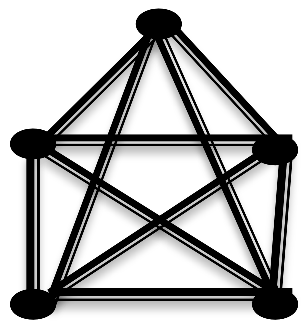
$\delta(F) = 1$

Burr–Erdos–Lovasz Theorem (Extension)

- Generally, for any graph H there exists a graph F' such that $F' \not\to H$ and the colors of some portion of F' are fixed however* we want.
- How does this help?
- We can create whatever* two coloring of a graph we want, then stick a vertex onto this graph.
- Argue that no matter which way we color the "stuck on" edges, we will always get a monochromatic copy of H.

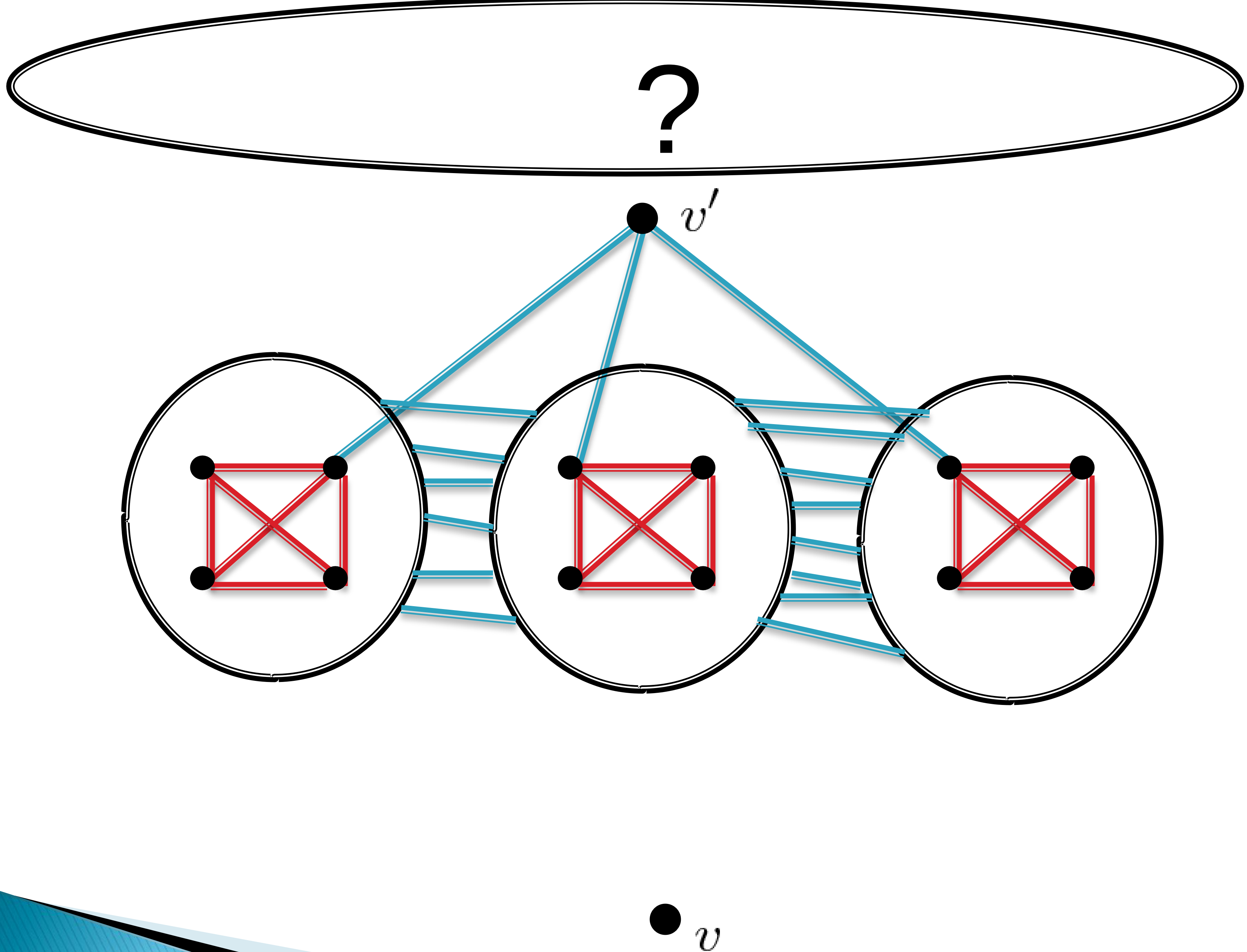
Complete Graph with Missing Edge

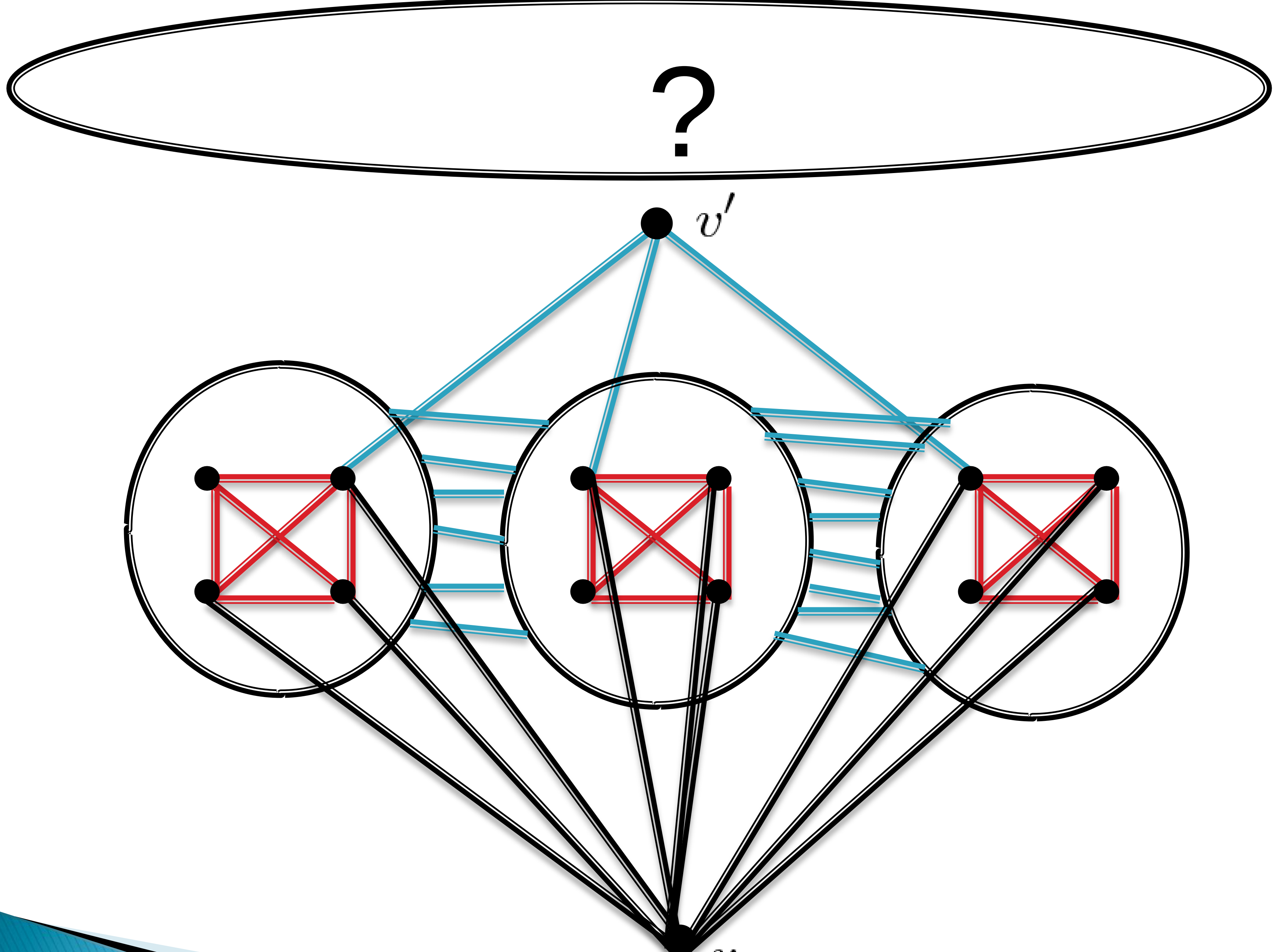
- Consider $K_5 - edge$ as an example

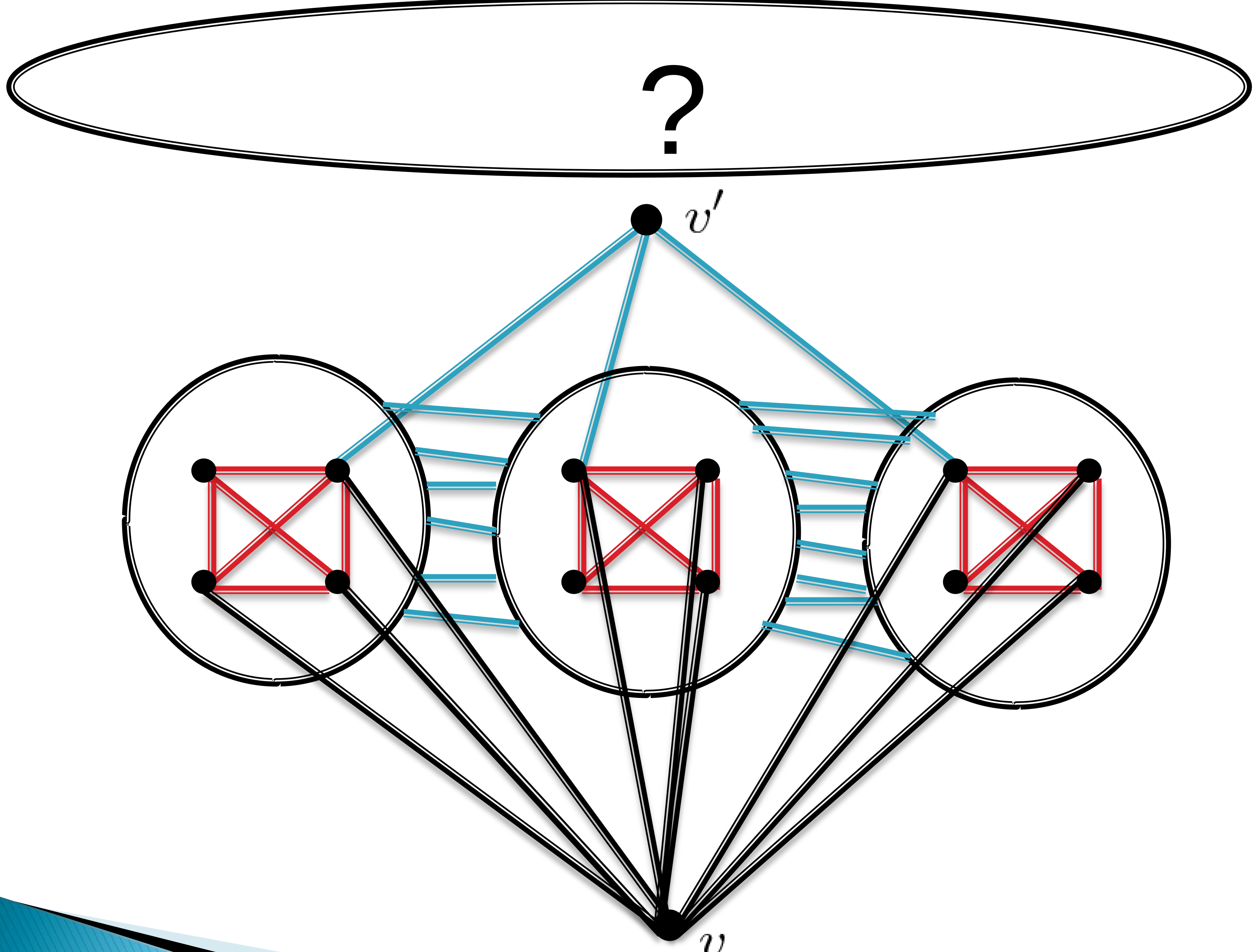


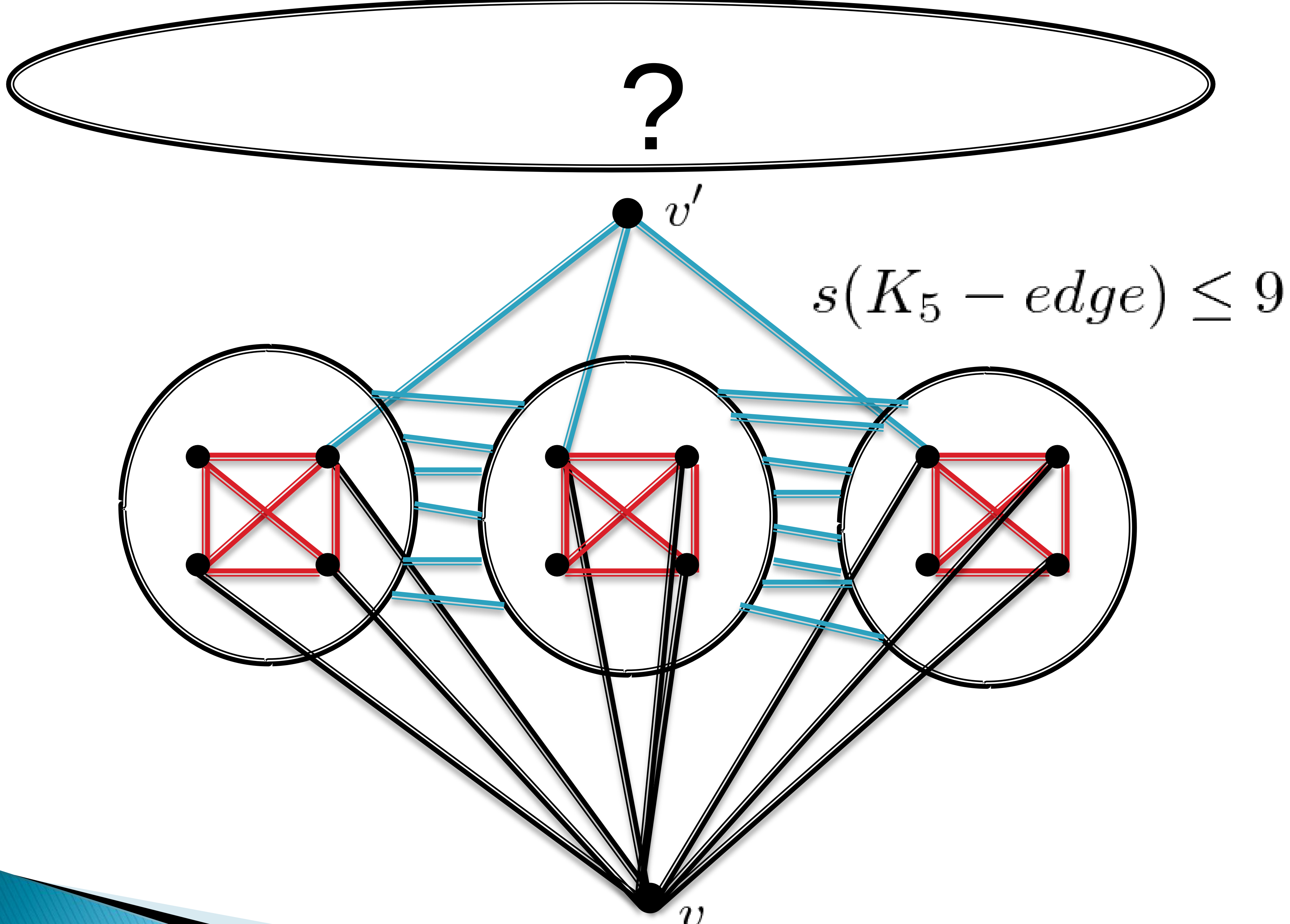
$K_5 - edge$

- Upper bound using Burr–Erdos–Lovasz.
- Apply the theorem to fix colors and stick a vertex onto it.


?
v'
v


?
v'
v


?
v'
v


?
v'
$s(K_5 - edge) \leq 9$
v

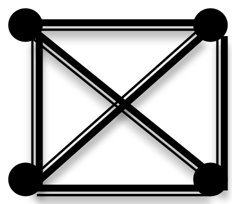
Lower Bound

- In general, take any minimal graph F with $F \to H$ and remove a vertex v of degree $\delta(F)$.
- Take any coloring of $F - v$ without a monochromatic copy of H and see what happens when you put v back in.

$$s(K_t - edge) = (t-2)^2$$

For the Future

- We proved that $s(K_{2t} - matching) \leq (t-1)(2t-1)$



$$s(K_4 - matching)$$

- What would be interesting would be to find $s(G(n,p))$