Towards generalizing thrackles to arbitrary graphs

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Thrackles and known results

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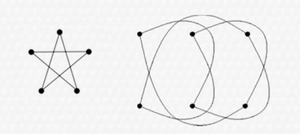

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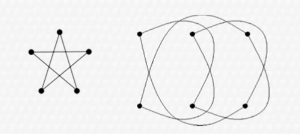

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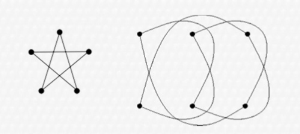

What is a graph that is not a thrackle?

C_4, the 4-cycle is not a thrackle. Let's see why.

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Theorem

*If G is a **linear thrackle** (has a thrackle drawing using straight lines), then $|E(G)| \leq |V(G)|$.*

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- Lovász-Pach-Szegedy: $c \approx 3$.
- Cairns-Nikolayevsky: $c \approx 1.5$.
- Best known bound: $c \approx 1.428$.

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Conjecture (O.)

A thrackle G has chromatic number at most 3.

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- *First out of all embeddings of G, choose only the ones that maximize the number of pairs of edges that crosses exactly once.*
- *Then, out of the remaining embeddings of G, choose only the ones that maximize the number of pairs of edges that do not cross.*
- *Iterate the process by maximizing the number of pairs of edges that crosses $2, 3, 4, \cdots$ times.*

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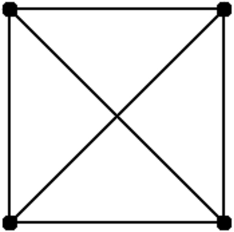

Let's see some more examples on the board.

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Conjecture (Weak Deletion Conjecture)

Suppose we have a near-thrackle drawing of a graph G. Then there exists ***some*** $v \in V(G)$ *such that deleting* v *from this drawing yields a near-thrackle drawing of* $G \setminus \{v\}$.

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Conjecture (Strong Deletion Conjecture)

Suppose we have a near-thrackle drawing of a graph G. Pick ***any*** *$v \in V(G)$, and delete v from that drawing. Then this is a near-thrackle drawing of $G \setminus \{v\}$.*

Near-thrackle drawings

Conjecture

A near-thrackle drawing of K_n is obtained by taking the n vertices in convex position, and then drawing all possible edges between them. In fact, this is the unique near-thrackle drawing of K_n up to small perturbations that do not disturb the convexity.

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A near-thrackle drawing of $K_{m,n}$ is obtained by taking $m + n$ vertices in convex position, and then defining m contiguous ones as one side of the partition, the n others as the other side of the partition, and drawing all possible edges between them. In fact, this is the unique near-thrackle drawing of $K_{m,n}$ up to small perturbations that do not disturb the convexity or ordering.

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Corollary

A near-thrackle drawing of K_n has $n(n-1)(n-2)(n+9)/24$ pairs of edges that cross exactly once, and the remaining pairs do not cross at all.

Thanks!

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- My parents

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- The Stony Brook School

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- Rik Sengupta
- The Stony Brook School
- Dr. Pavel Etingof, Dr. Ben Elias, and All PRIMES staff