The Effect of Inequalities on Partition Regularity of Linear Homogenous Equations

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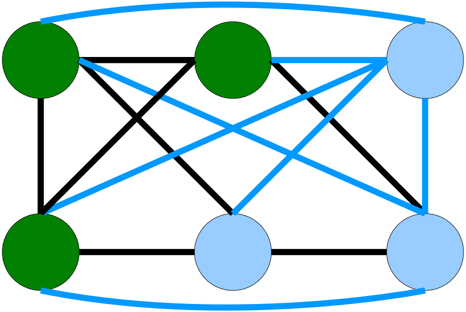
MIT-PRIMES

May 18, 2013

A Simple Example of Ramsey Theory: Six People

- Given 6 people, any 2 of whom are either friends or enemies
- Property: There always exists a group of 3, all of whom are friends or enemies with each other
- Not true with 5 people, so 6 is the minimum number that satisfies the property.



Definitions

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A **linear homogenous equation** is one of the form

$$c_0x_0 + c_1x_1 + c_2x_2 + \ldots + c_{n-1}x_{n-1} = 0, c_i \in \mathbb{Z}, x_i \in \mathbb{N}$$

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Definition

A linear homogenous equation of n variables is **regular** over the integers if, for any finite coloring of the natural numbers, there exist natural numbers $x_0, x_1, \ldots, x_{n-1}$ that:

- satisfy the equation and
- are the same color (are **monochromatic**).

Example: $x_0 + x_1 - x_2 = 0$.

1 2 3 4 5 6 7 8 9 10 11 . . .

Rado's Theorem

Given a **linear homogenous equation**:

$$c_0x_0 + c_1x_1 + c_2x_2 + \ldots + c_{n-1}x_{n-1} = 0, c_i \in \mathbb{Z}, x_i \in \mathbb{N}$$

where $c_0, c_1, \ldots, c_{n-1} \neq 0$,

Rado's theorem: states that this is regular if and only if some subset of c_i sum to 0.

$\mathbf{3}x_0 + 4x_1 - \mathbf{5}x_2 + \mathbf{2}x_3 = 0$ is regular, for example.

We extend this by considering the effect on regularity of a finite number of inequalities.

Inequalities do not affect regularity

Theorem

For $n, k, r \in \mathbb{N}$, and any r-coloring of the positive integers, there exists a monochromatic solution of the form $(x_0, \cdots, x_{n-1})$ to

$$\begin{aligned} \sum_{i=0}^{n-1} c_i x_i &= 0 : c_i \neq 0 \\ \sum_{i=0}^{n-1} A_{ji} x_i &\neq 0 : 1 \leq j \leq k \end{aligned}$$

where a nonempty set of the c_i sums to 0 and $A_{j0}x_0 + \cdots + A_{j(n-1)}x_{n-1}$ is not a multiple of $\sum_{i=0}^{n-1} c_i x_i$ for all $1 \leq j \leq k$.

A nonempty set of the c_i must sum to 0 by Rado's Theorem. $A_{j1}x_1 + \cdots + A_{jn}x_n$ not a multiple of $\sum_{i=1}^{n-1} c_i x_i$: if it were a multiple, it would always be 0.

Background Theorems for proof

- **Van der Waerden's theorem**: given any finite coloring of the integers, guarantees a monochromatic arithmetic progression of arbitrary length.
- Example: 1 [2] 3 4 [5] 6 7 [8] 9 . . .

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- Extension of Van der Waerden Theorem: guarantees a t-dimensional monochromatic arithmetic progression of arbitrary length:

$$a + d_1 l_1 + d_2 l_2 + \cdots + d_t l_t \; : \; 0 \leq l_i \leq L$$

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- $a = 1, t = 2, d_1 = 2(\text{horizontal}), d_2 = 3(\text{vertical}), L = 3$:

1	3	5	7
4	6	8	10
7	9	11	13
10	12	14	16

Strengthening Extended Van der Waerden's

Important lemma to prove theorem:

Lemma

For all $L \in \mathbb{N}$, a r-coloring of the natural numbers, $s_1, s_2, \ldots, s_t \geq 1$ and inequalities:

$$h_{j1}d_1 + h_{j2}d_2 + \cdots + h_{jt}d_t \neq 0$$

there exists $a, d_1, d_2, \ldots, d_t > 0$ such that

$$\left\{ a + \sum_{i=1}^{t} l_i d_i \; : \; 0 \leq l_i \leq L \right\} \cup \{ s_i d_i \; : \; 1 \leq i \leq t \}$$

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2 main ideas: Use extended Van der Waerden's Theorem, then

1. Add in inequalities on d_i.
2. Add in $\{s_i d_i\}$ that must be monochromatic.

- Goal is to find monochromatic x_i that satisfy $\sum c_i x_i = 0$ and $\sum A_{ji} x_i \neq 0$.
- Prove that there exist numbers of a certain form that are monochromatic.
- Prove that any numbers of this form satisfy the equation and inequalities.

The numbers of a "certain form" is the form we proved in the lemma:

$$\left\{ a + \sum_{i=1}^{t} l_i d_i \,:\, 0 \leq l_i \leq L \right\} \cup \{ s_i d_i \,:\, 1 \leq i \leq t \}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\{x_0, x_1, x_2, \ldots, x_{n-t-1}\} \qquad \{x_{n-t}, x_{n-t+1}, \ldots, x_{n-1}\}$$

With the parametrization shown above, it is simple to prove that numbers of this form satisfy the equation and inequalities.

Discussion

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Example: $x_0 - x_1 + 2x_2 + 4x_3 = 0$ is regular by Rado's Theorem.
1 2 3 4 5 6 7 8 9 10
$x_0 = x_2 = x_3 = 1, x_1 = 7$ is a monochromatic solution.
Say we want $x_2 - x_3 \neq 0$, though.

- So, add a finite number of inequalities.

Conclusion: Regularity does not change with inequalities.

Regularity

Definition

A linear homogenous equation is r-**regular** over $\mathbb{N}$ if, for every coloring of $\mathbb{N}$ with at most r colors, there exist monochromatic $x_0, x_1, \ldots, x_{n-1}$ that satisfy the equation.

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The **degree of regularity (**DoR**)** of a linear homogenous equation is the largest positive integer k such that it is k-regular.

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Definition

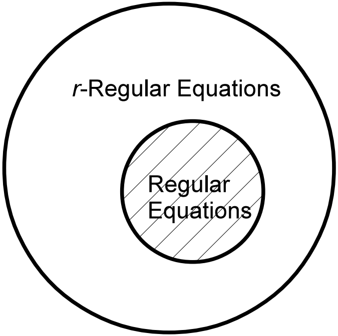
The **degree of regularity (**DoR**)** of a linear homogenous equation is the largest positive integer k such that it is k-regular.

Example:

- $-7x_0 + 6x_1 + 4x_2 = 0$ is 2-regular, but not 3-regular, so its $DoR = 2$

Goals

- We proved that inequalities do not affect regularity.
- What about non-regular but r-regular equations, for a given r?



Degree of Regularity

Family of equations:

$$\sum_{i=1}^{p-1} \frac{2^i}{2^i - 1} x_i = \left(-1 + \sum_{i=1}^{p-1} \frac{2^i}{2^i - 1} \right) x_0$$

Known result: for each value of p, the equation is known to have a degree of regularity of $p - 1$ (Alexeev & Tsimerman, 2009), verifying a conjecture of Rado.

We begin with small values of p.

$-7x_0 + 6x_1 + 4x_2 = 0$

- For $p = 3$, the equation simplifies to $-7x_0 + 6x_1 + 4x_2 = 0$.
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Theorem

Given a 2-coloring of the integers, the equation $-7x_0 + 6x_1 + 4x_2 = 0$, and a finite number l of inequalities:

$$A_{j0}x_0 + A_{j1}x_1 + A_{j2}x_2 \neq 0, 1 \leq j \leq l$$

none of which is a multiple of $-7x_0 + 6x_1 + 4x_2 = 0$, we can always find a monochromatic solution (x_0, x_1, x_2).

Use parametrizations!

- $(2n + 4k, n + 4k, 2n + k)$ is always a solution to $-7x_0 + 6x_1 + 4x_2 = 0$.
- We prove that there exist infinitely many pairs (n, k) that make the triplet monochromatic under any 2-coloring:
- Only a bounded number do not satisfy the inequalities

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Pick monochromatic AP: 4m, 4m+4d, 4m+8d, . . . , 4m + 4Ld
If there are no triplets of the desired form, these are blue:
m+8d, m+16d, . . . , m + 64d, m + 72d, . . . m + 4Ld
2m+16d, 2m+32d, 2m + 48d, 2m + 64d, . . . , 2m + 4Lk
But now set $m = n, k = 16d$, and we are done!

$p = 4$

Theorem

Given a 3-coloring of the integers, the equation

$$\sum_{i=1}^{3} \frac{2^i}{2^i - 1} x_i = \left(-1 + \sum_{i=1}^{3} \frac{2^i}{2^i - 1} \right) x_0$$

and a finite number l of inequalities: none of which is a multiple of the equation, we can always find a monochromatic solution (x_0, x_1, x_2, x_3).

To prove this, we again use parametrizations.

- $(2m + 4n + 8k, m + 4n + 8k, 2m + n + 8k, 2m + 4n + k)$ is always a solution to this equation.
- Proof to find a monochromatic quadruplet is somewhat similar.

Conditions on Regularity

Also explored *DoR* of various linear homogenous equations

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- First considered $a_0x_0 + a_1x_1 + a_2x_2 = 0$ in terms of its 2-regularity
- Used a similar approach as with $6x_0 + 4x_1 - 7x_2 = 0$

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Theorem

If, for some parametrization $(b_0y_0 + b_1y_1, b_0y_0 + b_2y_1, b_3y_0 + b_2y_1)$

- $b_0 \neq b_3,\ b_1 \neq b_2$
- $a_0b_0 + a_1b_0 + a_2b_3 = 0,\ a_0b_1 + a_1b_2 + a_2b_2 = 0$
- $b_0^2b_1 = b_3^2b_2$ *or* $b_1^2b_0 = b_2^2b_3$

then the equation $a_0x_0 + a_1x_1 + a_2x_2 = 0$ *is 2-regular. Furthermore, all 3 integers in the triple are not equal.*

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$-7x_0 + 6x_1 + 4x_2 = 0$ is a special case of this where $b_0 = 2, b_1 = 1, b_2 = 4, b_3 = 1$.

Tying this in with Inequalities

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- Show that a bounded number do not satisfy the inequalities

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Example: $-7x_0 + 6x_1 + 4x_2 = 0$

$$m + 8d, \ldots, \boxed{\text{m + 64d}}, m + 72d, \ldots, \boxed{\text{m+128d}}$$

$$\boxed{\text{2m+16d}}, \boxed{\text{2m+32d}}, 2m + 48d, \boxed{\text{2m + 64d}}, \ldots, \boxed{\text{2m+128d}}$$

If one triplet does not satisfy the inequalities, pick the other

Future Research

- Generalize the family of equations to any number of variables
- Analyze how inequalities affect the *DoR* of other linear homogenous equations
- What are the conditions on 3-regularity for an equation $a_0x_0 + a_1x_1 + \ldots + a_{n-1}x_{n-1} = 0$? 4-regularity? k-regularity?
- Generalize the idea that inequalities do not affect regularity to systems of linear homogenous equations
- Is a regular equation still regular if the x_i must be relatively prime?

Acknowledgements

Many thanks to:

- Laszlo Miklos Lovasz, our mentor
- Jacob Fox
- Tanya Khovanova
- Our awesome parents
- MIT PRIMES