# CONSTRUCTION OF THE HIGHER BRUHAT ORDERS ON THE WEYL GROUP OF TYPE B

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Abstract. Manin and Schechtman defined the Bruhat order on the type A Weyl group, which is closely associated to the Symmetric group $S_n$, as the order of all pairs of numbers in $\{1, 2, ..., n\}$. They proceeded to define a series of higher orders. Each higher order is an order on the subsets of $\{1, 2, ..., n\}$ of size $k$, and can be computed using an inductive argument. It is also possible to define each of these higher orders explicitly, and therefore know conclusively the lexicographic orders for all $k$. It is thought that a closely related concept of lexicographic order exists for the Weyl group of type B, and that a similar method can be used to compute this series of higher orders. The applicability of this method is demonstrated in the paper, and we are able to determine and characterize the higher Bruhat order explicitly for certain $n$ and $k$. We therefore conjecture the existence of such an order for all $n > k$, as well as its accompanying properties.

## 1. Introduction

A central object of study in Lie theory is the Weyl group. Weyl groups are finite groups which occur in several guises and with many important applications to representation theory. For example, to any semisimple Lie group $G$ one can define the Weyl group to be $W = N(T)/T$ for a maximal torus $T$ and its normalizer $N(T)$. Alternatively one can recover the Weyl group in a more concrete way as certain symmetries of a combinatorial object associated to $G$ called a root system.

A useful tool in representation theory is the *Bruhat decomposition* of $G$ into a finite set of locally closed pieces $G_w$ indexed by the elements of the Weyl group. Moreover, one can impose an order on $W$ called the *Bruhat order*, making $W$ into a ranked poset with a unique minimal element and a unique maximal one. This order respects the decomposition in the sense that the dimension of $G_w$ is the rank of $w$, and its closure contains all the $G_{w'}$ with $w' \leq w$. For an exposition of the basic properties of Weyl groups, see [H].

Weyl groups are part of a broader class of finite groups called *finite Coxeter groups*, groups generated by a finite set of reflections of Euclidean space. Much like the classification of simple Lie groups, finite Coxeter groups are classified into a finite list of "types." To any Coxeter group $W$ and generating set $S$ and

*Date*: May 27, 2014.

element $w \in W$ is associated a *reduced expression graph* $\tilde{\Gamma}_w$. This is an undirected graph whose vertices correspond to minimal length presentations of $w$ in terms of elements of $S$, and whose edges are given by applying a braid relation to transform one reduced expression into another. It was discovered that all of the loops in the reduced expression graph are built from loops in the reduced expression graphs of finite rank 3 Coxeter groups. Furthermore, there is a simplicial complex $A$ known as the Coxeter complex, whose largest simplices (i.e. chambers) correspond to $w \in W$; an expression in $S$ corresponds to a path through the walls of these chambers. For a precise discussion of the Coxeter complex and the reduced expression graph, we refer to [R].

In [MS], Manin and Schechtman give a different description of the Bruhat order, which generalizes to higher Bruhat orders. Let $C(n,k)$ denote the set of all subsets of cardinality $k$ in $\{1, 2, ..., n\}$. Certain orderings on $C(n,k)$ denote the set $A(n,k)$ of admissible orderings. One can define a rank function of admissible orderings, with unique minimal and maximal elements, $\rho_{\min}$ and $\rho_{\max}$. Furthermore, this rank function together with the notion of a "packet flip" gives the set $A(n,k)$ the structure of a ranked poset. When $k = 1$, this recovers the usual Bruhat order. Most importantly, there is a way to associate an element of $A(n,k+1)$ to a path from $\rho_{\min}$ to $\rho_{\max}$ in $A(n,k)$.

The purpose of this work is to generalize the higher Bruhat order of Manin and Schechtman to Weyl groups of other Lie types. This project is motivated by recent work of Ben Elias and Geordie Williamson. In [E], Elias recasts Manin and Schechtman's result in terms of Bott-Samelson bimodules. In this interpretation, reduced expressions for the maximal element of $W$ correspond to Bott-Samelson bimodules, and edges in the reduced expression graphs give morphisms between them (in either direction). In general, two different paths between the same reduced expressions will produce nonequal morphisms between Bott-Samelson bimodules. Elias finds that, in type A, the orientation defined by the higher Bruhat order from MS (and its opposite orientation) are the unique orientations for which two different oriented paths always produce the same morphism. We call such an orientation "compatible". In [Soergel calculus],[EW], Elias and Williamson demonstrate that in type $B_3$ there is also a unique (up to duals) compatible orientation, and in unpublished calculations, Elias has confirmed the existence of a compatible orientation for $B_n$ for $n$ small. This suggests that a compatible orientation on reduced expressions should exist for any type $B$ Coxeter group, and perhaps that the higher Bruhat orders should exist as well.

Also in [Soergel calculus], Elias and Williamson demonstrate that there is no compatible orientation in type $H_3$. Perhaps this is an indication that higher Bruhat orders are a feature unique to Weyl groups or crystallographic groups, but it is too early to make any conjectures.

Now let us describe the structure of this paper. First, we introduce some notation. In Chapter 2 we explore the case of the type A Weyl group, the recursive method for computing the higher Bruhat orders, and discuss the properties of this series of higher orders. Then, the notion of standard order, as the type B analog of lexicographic order in type A, is explored. A description of some additional notation is provided, and the *inductive process* of computing a higher order (in which $k = a+1$) from the previous order ($k = a$) is illustrated through the small cases of $k = 1, 2$, and 3. The general argument that serves as the inductive step remains valid for all $a \geq 1$. However, there are some additional subtleties to this inductive process for the first few Bruhat orders, which we will discuss in Chapter 7. For all $k \geq 3$, the exact same process of computation of the order on $C(n, k+1)$, given the order on $C(n, k)$ is valid. We define a certain set of orders as *good* orders. We then prove that it is sufficient for the Bruhat order on $C(n, k)$ to be good for the next higher order, $\rho_{\min} \in A(n, k+1)$ to be good. However, this proof requires us to show that the two aspects that define a higher order $r_1 \in A(n, k)$ with respect to an order $r_2 \in A(n, k)$ (namely, the concepts of the *parent* and *principal* aspects) both satisfy the properties claimed. From here, we conclude that the Standard order satisfies a recursion for all values of $n \geq 4$, and use this recursion to show the ultimate result of a description of the higher Bruhat order on the set $C(n, k)$ for all positive integers $n > k$. Therefore, we show that, given the existence of such an order in type B, it must satisfy all the properties we previously proved.

## 2. Background: type A Weyl group

The Weyl group of type A is defined by the Coxeter system $(W, S_n)$ where $S_n$ refers to the elementary transpositions $(1\ 2), (2\ 3), ..., (n-1\ n)$.

### 2.1. **Notation.**

**Notation 2.1.** *The set $I$ is the set of all integers from 1 to $n$ inclusive,* $\{1, 2, ..., n\}$.

**Notation 2.2.** *In Sections 2 and 3, the set $C(n, k)$ consists of all subsets of $I$ of cardinality $k$.*

**Definition 2.3.** *The* packet $P(K)$ *of* $K = (i_1, i_2, ..., i_{k+1}) \in C(n, k+1)$ *is the set of* $\{K_a^\wedge\}$ *for* $a = 1, ..., k+1$*, where* $K_a^\wedge = K - (i_a)$.

**Definition 2.4.** *The* lexicographic order *on the packet of* $K = (i_1, i_2, ..., i_{k+1})$ *(where* $i_1 < i_2 < ... < i_k$*) is the order* $K_{k+1}^\wedge < K_k^\wedge < ... < K_1^\wedge$*. Also, the* antilexicographic order *is defined as* $K_1^\wedge < K_2^\wedge < ... < K_{k+1}^\wedge$.

**Definition 2.5.** *An order $\rho$ on the elements of $C(n,k)$ is* admissible *if it induces lexicographic or antilexicographic order on the packet $P(K)$ for each $K \in C(n,k+1)$. The set of all such $\rho$ is denoted by $A(n,k)$.*

**Example 2.6.** *The order $\rho$, given by*

$$(2,3) < (1,3) < (1,2) < (1,4) < (2,4) < (3,4)$$

*induces lexicographic order on $P(1,2,4) = (1,2) < (1,4) < (2,4)$, $P(1,3,4) = (1,3) < (1,4) < (3,4)$, and $P(2,3,4) = (2,3) < (2,4) < (3,4)$, and antilexicographic order on $P(1,2,3) = (1,2) < (1,3) < (2,3)$. Thus $\rho \in A(3,2)$.*

**Definition 2.7.** *For an order $\rho \in A(n,k)$, we define $Inv(\rho)$ as the set of all $K \in C(n,k+1)$ such that $\rho$ induces antilexicographic order on $P(K)$. We let $inv(\rho) = |Inv(\rho)|$.*

**Definition 2.8.** *A total order $\rho$ is said to be* elementarily equivalent *to another order $\rho'$ if $\rho$ can be obtained from $\rho$ by switching pairs of elements $J_1, J_2 \in C(n,k)$ which are not members of a common packet (this is equivalent to $|J_1 \cup J_2| \geq k+2$) [MS].*

**Definition 2.9.** *For an order $\rho \in A(n,k)$, a* chain *in $\rho$ is an uninterrupted sequence of members of $C(n,k)$ comprising $P(K)$ for some $K \in C(n,k+1)$.*

**Notation 2.10.** *The order in which the chain of elements comprising $P(K)$ is inverted, but the rest of the elements retain their positions, is denoted by $p_K(\rho)$.*

**Example 2.11.** *For the order $r$ given by*

$$(2,3) < (1,3) < (1,2) < (1,4) < (2,4) < (3,4),$$

*we say $Inv(r) = (1,2,3)$; $r = p_K(\rho_{\min})$, where $K = (1,2,3)$.*

**Notation 2.12.** *For all such orders $r \in A(n,k)$, $inv(r)$ is a rank function on $A(n,k)$ that defines the structure of the equivalence class $B(n,k)$. Thus, the rank function* inv *descends to a function on $B(n,k)$, and every $r \in B(n,k)$ is uniquely defined by $Inv(r)$.*

**Definition 2.13.** *For $r \in B(n,k)$, the set of neighbors $N(r)$ contains all $K \in C(n,k+1)$ such that $P(K)$ forms a chain with respect to some $\rho \in r$.*

## 3. Previous research

In a 1989 publication, Manin and Schechtman defined the Bruhat order on the type A Weyl group, which is closely associated to the Symmetric group $S_n$, as the order of all pairs of numbers in $\{1, 2, ..., n\}$. Specifically, the Bruhat order on $C(n, 1)$ is the classical Bruhat order on the elements of the symmetric group $S_n$. Then $C(n, 2)$ could be interpreted as an order on reduced expressions for $w_0$, the longest element. Manin and Schechtman proceeded to define a series of higher orders, where each higher order is an order on the subsets of $\{1, 2, ..., n\}$ of size $k$, and can be computed using an inductive argument. Note that the higher Bruhat order is the partial order (as exhibited by the ranked poset $B(n, k)$ for all $k < n$) on the set of total orders.

It is also possible to define each of these higher orders explicitly, and therefore know conclusively the lexicographic orders for all $k$. First, it should be noted that for the case in which $k = 2$, the Bruhat order gives an order on the elements of the Symmetric group $S_n$, with respect to the simple generating set (which in this case is the set of transpositions).

Another critical result regarding type A was proved by Manin and Schechtman, which will be enumerated in the following three theorems (each pair of which are biconditional).

**Theorem 3.1.** *Given $r, r' \in B(n, k)$, we say that $r \leq r'$ if and only if there exist $K_i$ such that $K_i \in N(p_{K_{i-1}}...p_{K_1}(r)) - Inv(p_{K_{i-1}}...p_{K_1}(r))$, and $r' = p_{K_m}...p_{K_1}(r)$. This inequality gives a partial order on $B(n, k)$, and defines the structure of a ranked poset (rank function inv) which has minimal and maximal elements $r_{min} = \pi(\rho_{\min})$ and $r_{\max} = \pi(\rho_{max})$, respectively [MS].*

Thus, the admissible order constitute a set equivalence classes, ranging from $\rho_{\min}$ to $\rho_{\max}$. More specifically, the minimal element $\rho_{min}$ is the lexicographic ordering of inversions in $C(n, k)$ and the maximal element $\rho_{max}$ is the antilexicographic order. These classes are linked through the inversions of the packets $P(K)$ for $K \in C(n, k+1)$. Given two orderings, $r$ and $r'$, such that the poset $B(n, k)$ defines $r \geq r'$, $r'$ is the result when the appropriate packets are inverted in the order $r$. The order of this series of "packet flips" performed on $P(K)$ for $K \in C(n, k+1)$ is consistent with the higher order (that is, the order induced by the lexicographic order on $C(n, k+1)$).

**Theorem 3.2.** *The map from*

$$\{r_{\min} < p_{K_1}(r_{\min}) < ... < p_{K_m}...p_{K_1}(r_{\min})\} \to \rho = K_1...K_m$$

*defines a bijection from the*

$$\{the\ set\ of\ paths\ from\ r_{\min}\ to\ r_{\max}\} \to A(n, k+1)$$

*where* $r_{\min}, r_{\max} \in B(n,k)$*, and the paths are in accordance with the higher Bruhat order that defines the structure of the ranked poset on* $B(n,k)$*;* $A(n, k+1)$ *denotes the set of all admissible orders on the members of* $C(n, k+1)$ *[MS].*

One such series of inversions K, which transform the order on $C(n,k)$ from lexicographic to antilexicographic, is precisely the lexicographic order $\rho_{\min} \in A(n, k+1)$.

**Theorem 3.3.** *Every element* $r \in B(n,k)$ *is uniquely defined by the set* $Inv(r)$*: no two orders* $r, r' \in B(n,k)$ *can be the same if there exists* $K \in C(n, k+1)$ *such that* $P(K) \in Inv(r')$ *but* $P(K) \notin Inv(r)$*; also, two orders that have the same set of inversions are identical (modulo elementary equivalencies) [MS].*

**Lemma 3.4.** $A(n, n-1) = B(n, n-1) = \{K_n...K_1, K_1...K_n\}$ *where* $K = (1, ..., n)$ *[MS].*

That is, in Type A, the lexicographic order on $C(n, n-1)$ is precisely the packet of $(1, ..., n)$; the only other admissible order on $C(n, n-1)$ is antilexicographic.

3.1. **Explicit description of Lexicographic order.** From the above statements, lexicographic order on $(n,k)$ can be explicitly described as follows: For $i_1 = (a_1, ..., a_k)$, $i_2 = (b_1, ..., b_k)$, we have $i_1 < i_2$ if and only if $a_1 < b_1$ or $a_i = b_i$ for $i < m$ and $a_m < b_m$ for some $m \leq k$ [MS].

In type A, Manin and Schechtman defined the Bruhat order on the Weyl group (closely associated to the symmetric group $S_n$), and showed that it is possible to compute such a series of higher orders: Beginning with all total orders on the members of $C(n,1)$, we can consider the *classical Bruhat order* (a partial order on these total orders), in which the *lexicographic* element is defined as the total order $1 < 2 < ... < n$ (akin to the identity permutation), and the *antilexicographic* element is defined as the total order $n < n-1... < 1$. The aforementioned partially ordered set illustrates the *higher Bruhat order* on $C(n,1)$, and the oriented edges define paths from lexicographic to antilexicographic elements. Each of these oriented edges corresponds to an inversion of a member of $C(n,2)$, and so each path corresponds to a total order on $C(n,2)$. From these new total orders (including a lexicographic and antilexicographic) we can similarly define another higher Bruhat order, now on $C(n,2)$. This process of computing higher Bruhat orders can be continued until we reach the higher Bruhat order on $C(n,k)$, where $k = n-1$.

We claim that it is possible to similarly define a series of higher Bruhat orders on the Weyl group of type B. If such an order does exist, we conjecture that it can be constructed in a similar fashion to that discussed above.

## 4. Type B Weyl group

$W_{B_n}$ is the group of permutations generated by matrices symmetric across the skew diagonal. The Type B Weyl group $W_{B_n}$ is a subgroup of the symmetric group $S_{2n}$. Specifically, it is a subgroup of $2n$ by $2n$ permutation matrices. The lexicographic order on $C(n,1)$ can be thought of as the Coxeter group $(W,S)$ on the simple system $S$ (the set of transpositions $(i\ i+1)$ of generators. Thus $W_{B_n}$ can be thought of as symmetric permutations $\sigma$ of $\{\pm 1, ..., \pm n\}$, with generators $(i, i+1), (-i, -i-1)$ as well as $(-1, +1)$. For all $1 \leq i \leq n$, $\sigma(-i) = -\sigma(i)$. This gives rise to the equivalence classes which we will discuss later.

## 5. Notation in type B

**Notation 5.1.** *The set $I$ is the set of all nonzero integers from $-n$ to $+n$ inclusive,* $\{-n, ..., -1, +1, ..., +n\}$.

We begin by clarifying the notation that will be used in subsequent sections to describe the Weyl group of type B. First, $C(n,k)$ denotes a set of subsets of $I$, and members of $C(n,k)$ satisfy certain properties.

**Notation 5.2.** $C(n,1)$ *includes all subsets of $I$ of size* 1.

**Notation 5.3.** $C(n,2)$ *includes all subsets of $I$ of size* 2.

**Remark 5.4.** *Until otherwise specified, $J$ refers to a general member of $C(n,k)$: $J = (i_1, ..., i_b)$ which is a subset of $I$.*

**Notation 5.5.** *For a positive integer $m$ such that there exist either one or two values of $r$, $1 \leq r \leq b$, for which $m = |i_r|$, then $J_m^\wedge = J - (m) = (i_1, ..., i_b) - \{(i_r)\}$.*

**Example 5.6.** *For* $J = (-5, -1, +4)$, $J_1^\wedge = (-5, -1, +4) - \{(-1)\} = (-5, +4)$

**Example 5.7.** *For* $J = (-3, -2, +2, +3)$, $J_3^\wedge = (-3, -2, +2, +3) - \{(-3), (+3)\} = (-2, +2)$

**Remark 5.8.** *For $k \geq 2$, the members of $C(n,k)$ can be partitioned into a set of equivalence classes. For a set of $a_i$ of integers where $|a_i| \leq n$ for all $1 \leq i \leq b$, $(a_1, ..., a_b)$ and $(-a_1, ..., -a_b)$ are both members of $C(n,k)$ and share an equivalence class (which contains no other elements).*

**Definition 5.9.** *The two members* $J = (a_1, ..., a_b)$ *and* $-J = (-a_1, ..., -a_b)$ *sharing an equivalence class are said to be* complements *of each other, as* $J \cap -J = \emptyset$*. The member* $J$ *is said to be the* representative element *of the complement pair if and only if* $i_1 < 0$*.*

**Remark 5.10.** *From now on,* $J$ *generally refers to a representative element of* $C(n, k)$*.*

For subsets of the form $(-a, +a) \in C(n, 2)$, there are no other members in the equivalence class containing $(-a, +a)$. Note that $C(n, 2)$ can be thought of as the set of all $J$ of cardinality 2 such that $J^{\wedge}_{|i_1|}, J^{\wedge}_{|i_2|} \in C(n, 1)$. We therefore define subsequent sets $C(n, k)$ in a similar fashion: For $k \geq 2$, every $J \in C(n, k)$ satisfies $J^{\wedge}_{|i_1|}, ..., J^{\wedge}_{|i_a|} \in C(n, k - 1)$.

**Definition 5.11.** *The* packet $P(J)$ *is the set of all* $J' \in C(n, k - 1)$ *such that* $J' \subset J$ *or* $-J' \subset J$*.*

Note that $P(J)$ in some cases does not solely consist of the aforementioned set $J^{\wedge}_i$, although $P(J)$ must contain $J^{\wedge}_{|i_c|} \in C(n, k - 1)$ for all $1 \leq c \leq b$. Therefore, the members $J \in C(n, k)$ can be partitioned into two sets, which we will denote as $C_A(n, k)$ (those equivalence classes containing two members) and $C_B(n, k)$ (those equivalence classes containing one member). More precisely, all $J \in C_A(n, k)$ are of the form $(i_1, ..., i_k)$ (where each $i \in I$ is distinct), while all $J \in C_B(n, k)$ are of the form $(-j_1, ..., -j_{k-1}, +j_{k-1}, ..., +j_1)$ (where each $j \in \{1, ..., n\}$ is distinct). To recap, $C(n, k) = C_A(n, k) \cup C_B(n, k)$, where

$$C_A(n, k) = \{k - element\ subsets\ S\ of\ I :\ S \cap -S = \emptyset\}$$

modulo equivalence (i.e., $S\ \ -S$), and

$$C_B(n, k) = \{2(k - 1) - element\ subsets\ S\ of\ I :\ S = -S\}.$$

**Notation 5.12.** *For the sake of convenience, we will refer to every* $J \in C_B(n, k)$ *simply as* $(-j_1, ..., -j_{k-1}, 0)$*.*

Then the packet of $J$ for $J \in C_B(n, k)$ is $P(J) = \{(-i_1, \pm i_2, ..., \pm i_k)\} \cup \{J^{\wedge}_{j_1}, ..., J^{\wedge}_{j_{k-1}}\}$.

**Example 5.13.** *The packet of* $(-4, +3, +1, -2)$ *is*

$$\{(-4, +3, +1), (-4, +3, -2), (-4, +1, -2), (+3, +1, -2)\}$$

**Example 5.14.** *The packet of* $(-4, -2, -1, 0)$ *is*

$$\{(-4, -2, -1), (-4, -2, +1), (-4, -1, +2), (-4, +1, +2)\} \cup \{(-4, -2, 0), (-4, -1, 0), (-2, -1, 0)\}\}$$

With respect to the notion of a *packet* $P(K)$ in the type B Weyl group, the notions of *elementary equivalence*, *neighbor*, *chain*, *inversion*, and $Inv(u)$ are analogous to those given in type A.

**Notation 5.15.** *By convention, we refer to any equivalence class containing* $J = (i_1, ..., i_k)$ *by its* representative element *(*$J$ *if* $i_1 < 0$ *and* $-J$ *if* $i_1 > 0$*). If we mention by itself a* $J \in C(n, k)$ *where* $i_1 > 0$*, we are actually referring to the representative of its equivalence class, which is in this case its complement (*$-J$*).*

5.1. **Parent and child orders.**

**Definition 5.16.** *For any* $J \in C(n, k)$ *such that* $i_1 = -n$*, the* child *of* $J$ *is* $J_n^\wedge$.

**Definition 5.17.** *A* parent *of* $J \in C(n-1, k-1)$ *is a member* $K \in C(n, k)$ *such that* $K_n^\wedge = J$.

Note that there are two parent inversions for a $J \in C_A$ and one parent inversion for $J \in C_B$.

**Notation 5.18.** *For* $J = (i_1, ..., i_{k-1}) \in C(n-1, k-1)$*, the operation* $*$ *is defined as follows:* $-n * J$ *refers to the member* $(-n, i_1, ..., i_{k-1}) \in C(n, k)$*, while* $-n * -J$ *refers to the the member* $(-n, -i_1, ..., -i_{k-1}) \in C(n, k)$.

**Definition 5.19.** *Given a total order* $\rho = J_1 ... J_M$ *where for each* $1 \le a \le M$*,* $J_a \in C(n, k)$ *and* $n \in J_a$*, the* child order $\rho_n^\wedge$ *is the order* $(J_1)_n^\wedge ... (J_M)_n^\wedge$*, in which all redundant members (e.g. adjacent complementary members) have been removed.*

**Definition 5.20.** *Given a total order* $\rho = J_1 ... J_M$ *where for each* $1 \le a \le M$*,* $J_a \in C(n-1, k-1)$*, the* parent order $\rho'$*, denoted* $-n * \rho$*, refers to any total order of* $\{-n * J_a\} \cup \{-n * -J_a\}$ *for which the child order of* $\rho'$ *is simply* $\rho$.

5.2. **Principal order.**

**Definition 5.21.** *With respect to a member* $J \in C(n-1, k-1)$*, the two parents* $-n * J$ *and* $-n * -J$ *are said to comprise a* conjugate pair.

**Definition 5.22.** *The* principal member *of a conjugate pair* $-n * J, -n * -J$ *(where* $i_1 < 0$*), is* $-n * -J$*. The* antiprincipal member *of the pair is* $-n * +J$.

**Definition 5.23.** *Similarly, for* $J = (i_1, ..., i_k)$ *where* $i_1 < 0$*, the pair* $-n * -J, -n * +J$ *is said to be in* positive *principal order, while the pair* $-n * +J, -n * -J$ *is said to be in* negative *principal order.*

For the purposes of describing the principal order of a total order, we will use the symbols + and −, to denote positive and negative principal orders of the pairs within $C_A$ (the members of $C_B$ are disregarded when describing the principal order, as they do not form conjugate pairs).

## 6. Standard Order in Type B

We now wish to consider the analog in the type B case of the higher Bruhat order on the Weyl group of type A. In doing so, we seek an order analogous in the type B Weyl group to the lexicographic order in the type A Weyl group. We will call such an order the *Standard order* in type B. Note that in type A, the higher bruhat order on the equivalence classes of admissible total orders of $C(n, 1)$ is the classical bruhat order on the elements of the symmetric group $S_n$. Because the Weyl group of type B is given by the Coxeter system $(W, S_{2n})$, we define the standard order $S(n, 1)$ as the identity permutation:

$$-n < ... < -1 < +1 < ... < +n.$$

**Notation 6.1.** *$S(n, k)$ denotes the standard order, commonly referred to as $\rho_{\min}$, which is a total order on the elements of $C(n, k)$.*

**Notation 6.2.** *$S^{-1}(n, k)$, commonly referred to as $\rho_{\max}$, denotes the opposite order of the elements in $C(n, k)$ as that of $S(n, k)$.*

## 7. Construction of standard order $S(n, k)$ for small k

### 7.1. k=1.

As stated previously, the *standard order* $S(n, 1)$ on $C(n, 1)$ is defined for convenience as the order

$$-n, -(n-1), ..., -1, +1, ..., +(n-1), +n$$

In the following section, we will show how this order can be computed for a larger value of $k$, and indicate the properties of the standard order.

### 7.2. k=2.

**Theorem 7.1.** *For two distinct inversions $J_1 = (a_1, a_2) \in C(n, 2)$ and $J_2 = (b_1, b_2) \in C(n, 2)$, $J_1 < J_2$ if either $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$. Otherwise, $J_1 > J_2$.*

**Proposition 7.2.** *The standard order $S(n,2)$ on $C(n,2)$ for $n \geq 2$ reduces to the classical Bruhat order in the Type B Weyl group. So given $S(n-1,1) = J_1...J_{2M}$ for $2M = |C(n,2)|$, we claim the order $S(n,2) = (-n * J_1)...(-n * J_M); (-n,0); (-n * J_{M+1})...(-n * J_{2M}); S(n-1,2)$.*

As a base case, one can easily see that the permutation $\sigma = \{1,-1\}$ of $\{-1,1\}$ requires only the inversion $(-1,0)$. Next, consider the set

$$(-m, -(m-1), ..., -1, +1..., +(m-1), +m).$$

The permutation (-m +m) arises from the series of transpositions

$$(-m\ \ -(m-1))(-(m-1)\ \ -(m-2))...(m-2\ m-1)(m-1\ m),$$

which corresponds to performing the series of inversions

$$(-m, -(m-1)), (-m, -(m-2)), ..., (-m, -1), (-m, 0), (-m, +1), ..., (-m, +(m-2)), (-m, +(m-1)).$$

Thus, by the inductive hypothesis, the permutation

$$\{+m, +(m-1), ..., +1, -1, ..., -(m-1), -m\}$$

is given by the inversions

$$(-m * S(m-1,1)); S(m-1,2)$$

where $-m * S(m-1,1)$ is an order. (In general, note that $-n * S(n-1,k-1)$ is an order containing each parent of each member of $S(n-1,k-1)$, and whose child order is $S(n-1,k-1)$.) Note also that for $a > b > 0$, $(-a,-b)$ precedes $(-a,+b)$, and so each of the conjugate pairs within $-m * S(m-1,1)$ have negative principal order. Note that members of conjugate pairs are not adjacent to each other. We can then make the following statement regarding the standard order:

$$S(n,2) = [\prod_{i=-(n-1)}^{+(n-1)} (-n,i)]; S(n-1,2)$$

Note that the non-recursive description of the standard order offered by theorem 7.1 follows directly from this statement.

7.3. **Split order.**

**Definition 7.3.** *A total order* $\rho$ *of the elements in* $C(n,k)$ *is called* split *if it can be written with left, middle, and right parts* $\sigma_1, \tau$, *and* $\sigma_2$, *respectively, where* $\sigma_1\sigma_2 = S(n-1,k)$ *and* $\tau = -n * \pm rho'$ *for some total order* $\rho'$ *on the elements of* $C(n-1,k-1)$. *Alternatively, this means that a split order can be written in the following way:*

$$\rho = J_1...J_a; -n * (J_1'...J_M'); J_{a+1}...J_N$$

*(or* $-n * (J_1'...J_M'); J_1...J_N$ *if there is no left part;* $J_1...J_N; (J_1'...J_N')$ *if there is no right part) where* $J_i' \in C(n-1,k-1)$ *for all* $1 \leq i \leq M$, $J_j \in C(n-1,k)$ *for all* $1 \leq j \leq N$.

**Theorem 7.4.** *Given a split order* $\rho$ *such that it is possible to invert the packet* $P(-n*+J_a)$ *or* $P(-n*-J_a)$, *we can invert both of these packets in sequence to obtain another split order.*

*Proof.* Begin with a split order, $\rho$ which has middle part $-n * (J_1'...J_M')$ and right part $J_a...J_N$. (Note that $J_a$ is therefore the first member of the right part.) The union of packets $P(-n * -J_a)$ and $P(-n * -J_a)$ is the set consisting of the inversion $J_a$ and the parents of the packet $P(J_a)$; the intersection of the packets $P(-n * -J_a)$ and $P(-n * -J_a)$ is the set containing only $J_a$. Therefore, if we are initially able to invert the members of $P(-n * -J_a)$ (or respectively $P(-n * +J_a)$), it is because the parents of $P(J_a)$ which are members of $P(-n * -J)$ (resp. $P(-n * +J)$) form a chain within the middle part of $\rho$. The middle part of $\rho$ is composed only of adjacent conjugate pairs $-n * -J$ and $-n * +J$ for each $J \in C(n-1,k-1)$, and furthermore all $(-n * J') \in P(-n * -J_a)$ (resp. $(-n * J') \in P(-n * +J_a)$) are second in their respective conjugate pairs. Thus, the inversion of $P(-n * -J_a)$ (resp. $P(-n * +J_a)$) results in an order in which $J_a$ now resides between the other members of $P(-n * +J_a)$ (on the left, unchanged in order) and $P(-n * -J_a)$ (on the right, reversed in order). Since the order of other members within $P(-n * +J_a)$ is unchanged, the current order can also accommodate the inversion of this packet. Next, since $J_a$ is now to the left of the other members of $P(-n * +J_a)$ and $P(-n * -J_a)$, and the rest of the middle part of $\rho$ is disjoint with $J_a$, we can now move $J_a$ to the left part of $\rho$. Finally, it's possible to reconcile each of the conjugate pairs from $P(-n * +J_a) \cup P(-n * -J_a)$ so that the members of each pair are adjacent once again. The resulting order is, therefore, a split order. □

**Notation 7.5.** *For* $J \in C(n-1,k)$, *the notation* $-n * \pm J$ *refers to the parent inversion(s) to* $J$. *Thus, for* $J \in C_A(n-1,k)$, $-n * \pm J$ *denotes the union* $-n * -J, -n * +J$. *For* $J \in C_B(n-1,k)$, $-n * \pm J$ *denotes* $-n * J$.

**Lemma 7.6.** *Two inversions* $(-n * +K_a^\wedge)$ *and* $(-n * -K_b^\wedge)$ *commute if* $a \neq b$.

*Proof.* Two inversions $J_1, J_2 \in C(n, k-1)$ commute if they are not part of a common packet $P(K) \subset C(n,k)$. Since any such $K$ has $k$ elements, two inversions with a total of $k+1$ or more distinct elements cannot be contained within the same packet. In the case of the two inversions in question, $|(-n* +K_r^{\wedge}) \cup (-n* -K_s^{\wedge})| = |\{-n, +i_1, ..., +i_{r-1}, +i_{r+1}, ..., +i_{k-1}\} \cup \{+n, +i_1, ..., +i_{s-1}, +i_{s+1}, ..., i_{k-1}\}| = |\{-n, +n, +i_1, ..., +i_{k-1}\}| = k+1$. Thus, two such inversions commute. □

**Theorem 7.7.** *Given a split order $\rho$ such that either the members of $P(-n*+J_{a+1}) = -n*+(J_{a+1})^{\wedge}_{i_{k-1}}, ..., -n* +(J_{a+1})^{\wedge}_{i_1}, J$ or the members of $P(-n*-J_{a+1}) = -n*-(J_{a+1})^{\wedge}_{i_{k-1}}, ..., -n*-(J_{a+1})^{\wedge}_{i_1}, J$ form a chain with respect to $\rho$, we can invert both of these packets in sequence to obtain another order which is also split.*

*Proof.* Case 1 (possible to invert $-n* +J_{a+1}$): the initial split order $\rho$ must be elementarily equivalent to an order in which the members of $P(-n* J_{a+1})$ form a chain. Note that all conjugate pairs are grouped together, and so the only possible good order satisfying this condition induces the following order on $P(-n* \pm J_{a+1})$: $-n*-(J_{a+1})^{\wedge}_{i_{k-1}}, -n*+(J_{a+1})^{\wedge}_{i_{k-1}}, ..., -n*-(J_{a+1})^{\wedge}_{i_1}, -n*+(J_{a+1})^{\wedge}_{i_1}, J_{a+1}$. By lemma 7.5, this is elementarily equivalent to $-n*-(J_{a+1})^{\wedge}_{i_{k-1}} ... -n*-(J_{a+1})^{\wedge}_{i_1}, -n*+(J_{a+1})^{\wedge}_{i_{k-1}}, ..., -n*+(J_{a+1})^{\wedge}_{i_1}, J_{a+1}$. Thus, $p_{-n*+J_{a+1}}(\rho) = -n*-(J_{a+1})^{\wedge}_{i_{k-1}} ... -n*-(J_{a+1})^{\wedge}_{i_1}, J_{a+1}, -n*+(J_{a+1})^{\wedge}_{i_1}, ..., -n*+(J_{a+1})^{\wedge}_{i_{k-1}}$. In this resulting order, the members of $P(-n*-J_{a+1})$ form a chain, and so the inversion of this packet results in the order $p_{-n*-J_{a+1}}p_{-n*+J_{a+1}}(\rho)$ which induces the following order on $-n* \pm J_{a+1}$:

$$J_{a+1}, -n*-(J_{a+1})^{\wedge}_{i_1}, -n*+(J_{a+1})^{\wedge}_{i_1}, ..., -n*-(J_{a+1})^{\wedge}_{i_{k-1}}, -n*+(J_{a+1})^{\wedge}_{i_{k-1}}.$$

Note that this order is elementarily equivalent to $-n*-(J_{a+1})^{\wedge}_{i_1}, -n*+(J_{a+1})^{\wedge}_{i_1}, ..., -n*-(J_{a+1})^{\wedge}_{i_{k-1}}, -n* +(J_{a+1})^{\wedge}_{i_{k-1}}$. By lemma 7.6, $J_{a+1}$ is disjoint with the other members of the middle part, so it can be pushed up to the left part of the split order. Note that the left part of the split order is now $J_1...J_{a+1}$, the right part is $J_{a+2}, ..., J_N$, and the middle part is simply the parent order of $p_{J_{a+1}}(\rho)$. Therefore, the resulting order is $J_1...J_{a+1}; p_{J_{a+1}}(\rho); J_{a+2}, ..., J_N$, which is also split.

Case 2 (possible to invert $-n*-J_{a+1}$): note that this case is similar to the previous one, except that we first invert $P(-n*-J_{a+1})$ and then $P(-n*+J_{a+1})$, to obtain the resulting order $J_1...J_{a+1}; p_{J_{a+1}}(\rho); J_{a+2}, ..., J_N$.

Case 3: Given a $J_{a+1} \in C_B(n-1, k-1)$, there is only the single parent packet $P(-n* J_{a+1})$. We begin with this packet in standard order. Note that there is only one member of the packet (namely $J_{a+1}$ which does not belong to the middle part of the split order. □

By thm. 7.7, we can continue this process, inverting through the parents of $J_1, ..., J_N$ until we reach the order $J_1, ..., J_N; -n * (J'_M, ..., J'_1)$. Then we need only invert the packets within the left part, $J_1, ..., J_N$, in order to have completely reversed the initial order, and get the order $J_N, ..., J_1; J'_M, ..., J'_1$.

7.4. **k=3.** We begin with the order $S(n,2) = -n * S(n-1,1); S(n-1,2)$ and we wish to find the set of inversions that will transform this order to $S^{-1}(n,2) = S^{-1}(n-1,2); -n * S^{-1}(n-1,1)$ (this order of inversions is the same as the order on $S(n,3)$). Therefore, we claim that the series $\sigma_a = -n*(-a*S(a-1,1))$ of inversions for $a = n-1, n-2, ..., 1$ is such an order.

*Proof.* We will show by induction that it is possible to perform such a series of inversions in steps, inverting members of $\sigma_{n-1}$, of $\sigma_{n-2}$,...,$\sigma_2$, and a finally inverting $(-n,-1,0)$.

**Inductive Hypothesis 7.8.** *After the inversion of the members of a series $\sigma_a$ for some $a \geq 2$, the order on the members of $C(n,2)$ is*

$$-(n-1)*S(n-2,1), ..., -a*S(a-1,1); (-n,+(n-1)), ..., (-n,+a); -n*S(a-1,1); (-n,-a), ..., (-n,-(n-1)); S(a-1,2).$$

Base case: $\rho_{\min} = S(n,2) = -n*S(n-1,1); S(n-1,2)$, so the members of $\sigma_{n-1} = -n*(-(n-1)*S(n-2,1))$ are the leftmost members of the right part of $\rho_{\min}$. As such, it is possible to invert these members of a split order. This yields the order $-(n-1)*S(n-2,1); (-n,+(n-1)); -n*S(n-2,1); (-n,-(n-1)); S(n-2,2)$, as predicted by the inductive hypothesis for $a = n-1$. After inverting the members in some of these series $\sigma_{n-1}, ..., \sigma_{b+1}$ for some $b > 2$, assume that the resulting order is that stated in the hypothesis. Then we next invert the members of $\sigma_b$, and obtain

$$-(n-1)*S(n-2,1), ..., -b*S(b-1,1); (-n,+(n-1)), ..., (-n,+b); -n*S(b-1,1); (-n,-b), ..., (-n,-(n-1)); S(b-1,2).$$

Therefore, after inverting $\sigma_2$, the resulting order is

$$-(n-1)*S(n-2,1), ..., -2*S(1,1); (-n,+(n-1)), ..., (-n,+2); -n*S(1,1); (-n,-2), ..., (-n,-(n-1)); (-1,0).$$

Now, if we simply invert the packet $P(-n,-1,0)$, the resulting order is elementarily equivalent to

$$S(n-1,2); -n * S^{-1}(n-1,k-1).$$

We can see that the left part of the order $S(n,3)$ is $\sigma_{n-1}, ..., \sigma_2; (-n,-1,0)$. □

Therefore, from the computations above, we obtain the following about the standard order:

$$S(n,3) = (-n * S(n-1,2)); S(n-1,3)$$

where for $J \in C(n-1,2)$, $(-n{*}{+}J)$ precedes $(-n{*}{-}J)$ if and only if $J = (+j_1, -j_2)$ for some $j_1, j_2 > 0$. (In other words, the principal order of each conjugate pair $(-n{*}{+}J), (-n{*}{-}J)$ is positive for all $J = (-j_1, -j_2)$, negative for all $J = (-j_1, +j_2)$.)

**Notation 7.9.** *Given an order $\rho$, the inversion of the two packets $P(-n * J)$ and $P(-n * -J)$ is denoted by $p_{-n*\pm J}(\rho)$.*

7.5. $k \geq 4$**.** Beginning with the first conjugate pair (which has positive principal order), the principal order of the conjugate pairs alternates between positive and negative.

7.6. **Recursion.**

**Theorem 7.10.** *For all $k \geq 2$, the standard order $S(n,k)$ is given by the following recursion:*

$$S(n,k) = (-n * S(n-1,k-1)); S(n-1,k)$$

*. For $k \geq 4$, the principal order of $S(n,k)$ is the repeating pattern $+,-,-,+$.*

**Lemma 7.11.** *The packets of two members $J_1, J_2 \in C(n,k)$ are said to be* disjoint *if their packets do not intersect ($P(J_1) \cap P(J_2) = \emptyset$). This is equivalent to $|J_1 \cup J_2| \geq k+2$.*

Proof: Let $J_1 = (i_1, .., i_k)$ and $J_2 = (j_1, ..., j_k)$ be two members of $C(n,k)$ whose packets are disjoint. Then if $J_1 \cup J_2 \leq k+1$ there exists some $a, b$, $1 \leq a, b \leq k$ such that $(J_1)^{\wedge}_{i_a} = (J_2)^{\wedge}_{j_b}$. But this is a contradiction, as we defined $J_1, J_2$ as having non-intersecting packets. Therefore, the given statements are equivalent.

## 8. Proof

**Assumption 8.1.** $S(n,k-1) = -n{*}S(n-1,k-2); S(n-1,k-1)$*,* $S(n,k) = -n{*}S(n-1,k-1); S(n-1,k-1)$ *(where in the latter case $-n*$ denotes a principal order of conjugate pairs with the repeating pattern of $+--+$)*

**Assumption 8.2.** *The principal order of $\rho_{\min}$ initially allows us to invert the packets of $-n * J_i$ and $-n * -J_i$ for all $J_i < (-(n-1), -(n-2), ..., -(n-k+2), 0)$.*

**Inductive Hypothesis 8.3.** *$S(n,a)$ is a split order for all $a \leq k$*

**Lemma 8.4.** *for order $\rho \in B(n,k)$ and $K \in C_A(n-1,k)$, inversion of $P(-n*K)$ and $P(-n*-K)$ (in whichever order is possible) (a) is the parent order of the $p_K(\rho_n^{\wedge})$, and (b) preserves principal status of each of the conjugate pairs*

8.1. **Inversions on $P(J)$ for $J \in C_A$.** In the case where $J \in C_A(n-1,k)$, there are two parents, namely $-n*J$ and $-n*-J$. Therefore, the packets of the parents contain: $P(-n*J) \cup P(-n*-J) = \{-n*J_{i_k}^{\wedge}, ..., -n*Ji_1{}^{\wedge}, J\} \cup \{-n*-J_{i_k}^{\wedge}, ..., -n*-Ji_1{}^{\wedge}, J\}$. Since by the inductive hypothesis the two members of each conjugate pair are adjacent, we know that the order induced by $\rho$ on $-n*\pm J$ is either $-n*J_{i_k}^{\wedge}, -n*-J_{i_k}^{\wedge}..., -n*Ji_1{}^{\wedge}, -n*-Ji_1{}^{\wedge}, J$ or $-n*-J_{i_k}^{\wedge}, -n*J_{i_k}^{\wedge}..., -n*-Ji_1{}^{\wedge}, -n*Ji_1{}^{\wedge}, J$.

Therefore, by lemma ? the former order is elementarily equivalent to $-n*J_{i_k}^{\wedge}, ..., -n*J_{i_1}^{\wedge}, -n*-J_{i_k}^{\wedge}, ..., -n*-J_{i_1}^{\wedge}, J$, and the latter order is elementarily equivalent to $-n*-J_{i_k}^{\wedge}, ..., -n*-J_{i_1}^{\wedge}, -n*J_{i_k}^{\wedge}, ..., -n*J_{i_1}^{\wedge}, J$. In the first case, by our hypothesis, the packet $P(-n*-J)$ is in standard order, and so we can invert it. $p_{-n*-J}(\rho)$ then induces the following order on $-n*\pm J$: $-n*J_{i_k}^{\wedge}, ..., -n*J_{i_1}^{\wedge}, J, -n*-J_{i_1}^{\wedge}, ..., -n*-J_{i_k}^{\wedge}$. Notice that this new order induces standard order on the packet $P(-n*J)$, and so we can invert through this packet to obtain $p_{J_{-n*J}} p_{J_{-n*-J}}(\rho)$, which now induces the order $J, -n*+J_{i_1}^{\wedge}, ..., -n*+J_{i_k}^{\wedge}, -n*-J_{i_1}^{\wedge}, ..., -n*-J_{i_k}^{\wedge}$ on $-n*\pm J$. But this order is elementarily equivalent to the following order: $-n*J_{i_1}^{\wedge}, -n*-J_{i_1}^{\wedge}, ..., -n*J_{i_k}^{\wedge}, -n*-J_{i_k}^{\wedge}, J$.
By similar reasoning, with the latter order, we can perform the inversions in the opposite order $(p_{J_{-n*-J}} p_{J_{-n*J}}(\rho)$, which yields the order $p_{-n*\pm J}(\rho) = -n*-J_{i_1}^{\wedge}, -n*J_{i_1}^{\wedge}, ..., -n*-J_{i_k}^{\wedge}, -n*J_{i_k}^{\wedge}, J$. Thus, for $J \in C_A(n-1,k)$, the child order induced on $p_{-n*\pm J}(\rho)$ by $-n*\pm J$ is the order induced on $p_J(\rho_n^{\wedge})$ by $J$.

8.2. **Inversions on $P(J)$ for $J \in C_B$.** In the case where $J \in C_B(n-1,k)$, there is only one parent inversion $P(-n*J)$, so the inversion of $p_{-n*\pm J}(\rho)$ simply inverts the order of all the members of the packet $P(-n*J)$. Thus, for $J \in C_B(n-1,k)$, the child order induced on $p_{-n*\pm J}(\rho)$ by $-n*\pm J$ is the order induced on $J$ by $p_J(\rho_n^{\wedge})$.

**Remark 8.5.** *To recap: From a total order $\rho$ on $C(n,k+1)$, inverting the parents of $J \in C(n-1,k)$, namely $-n*\pm J$, alters the principal status of the these members if and only if $J \in C_B$. Furthermore, $p_{-n*\pm J}(\rho)$ is always a parent order of $p_J(\rho_n^{\wedge})$.*

Thus, if we begin with the child order $\rho_{k-1} = S(n-1,k-1)$ and perform an inversion on it, the parents of this inversion can be performed on $\rho_k = S(n,k)$, resulting in a middle part who is the parent of the child order after the inversion. We can therefore continue this process, and find that, for each inversion performed on $\rho_{k-1}$, we can perform its parent on $\rho_k$. Then, if we begin with Standard order $\rho_{\min} = S(n,k) = (J_1...J_M);(J'_1...J'_N)$ and invert every packet $P(K)$ until the order becomes $\rho_{\max} = (J'_N...J'_1);(J_M...J_1)$, the series of $K$ gives the Higher Standard order $S(n,k+1)$, whose middle part is the parent order of $S(n-1,k)$.

**Corollary 8.6.** *The series of inversions $p_{-n*+K_m}p_{-n*-K_m}p_{-n*-K_{m-1}}p_{-n*+K_{m-1}}...p_{-n*-K_1}p_{-n*+K_1}(\rho)$ where $K_a \in C_A(n-1,k-1)$ for each $1 \leq a \leq m$, $\rho \in C(n,k)$ is the parent order of $p_{K_m}...p_{K_1}(\rho_n^{\wedge})$.*

8.3. **Statement of S(n,n-1).** The higher Bruhat order induces the following order for the packet of $k = n-1$, which is analogous to the order induced on the members of the packet, $P(-n,-(n-1),...,0)$:

$$\textit{Order induced on packet } = \begin{cases} J^{\wedge}_{i_k},...,J^{\wedge}_{i_1}, & \text{if } J \in C_A(n,k) \\ -n*\rho; J^{\wedge}_{j_1}; -n*\rho'; (-n,i_1,..,i_k)^{\wedge}_{i_1},..,(-n,i_1,..,i_k)^{\wedge}_{i_k} & \text{if } J \in C_B(n,k) \end{cases}$$

where $\rho$ is the order on all such $(i_1,...,i_{k-2},i_{k-1})$ where $|i_a| = j_a$, for which $i_{k-2},i_{k-1}$ have the same sign; $\rho'$ is the corresponding order on $(i_1,...,i_{k-2},-i_{k-1})$ for which $i_{k-2},i_{k-1}$ do not have the same sign.

## 9. Conjectures

The standard order on each $C(n,n-2)$ is not uniquely determined. However, once it is defined we have the standard order on $C(n,n-1)$. From here we can uniquely deduce $C(n+i,n-1)$ for any positive $i$.

**Theorem 9.1.** *For the elements of $C(n,n)$, we find that the first inversion is $J_1 = (-n\ +(n-1)\ -(n-2)\ +...+3\ -2\ -1)$ for $n$ even; $(-n\ +(n-1)\ -(n-2)\ +...-3\ +1\ +2)$ for $n$ odd. Furthermore, if the inversions $J_1,...,J_{2^{k-1}}$ have a common descendent $(i_1,\pm i_2,...,\pm i_k)$ of rank $k$, the inversion $J_{n-i+1}$ is the inversion $J_i$ with the signs of the $i_1,...,i_k$ flipped.*

## 10. Acknowledgments

I would like to thank my mentor, Daniel Thompson, for all of his enthusiasm, helpful support, and guidance. I would also like to thank Dr. Ben Elias for providing this research problem. Finally, I give thanks to the MIT PRIMES-USA program for an excellent research experience.