On an Extension of Stanley Depth for Refinement-Ordered Posets

Ying Gao *

December 30, 2013

Abstract

The concept of Stanley depth was originally defined for graded modules over commutative rings in 1982 by Richard P. Stanley. However, in 2009 Herzog, Vladiou, and Zheng found a property, ndepth, of posets analogous to the Stanley depths of certain modules, which provides an important link between combinatorics and commutative algebra. Due to this link, there arises the question of what this ndepth is for certain classes of posets.

Because ndepth was only recently defined, much remains to be discovered about it. In 2009, Biro, Howard, Keller, Trotter and Young found a lower bound for the ndepth of the poset of nonempty subsets of $\{1, 2, ..., n\}$ ordered by inclusion. In 2010, Wang calculated the ndepth of the product of chains $n^k \setminus 0$. However, ndepth has yet to be studied in relation to many other commonly found classes of posets. We chose to research the properties of the ndepths of one such well-known class of posets - the posets which consist of non-empty partitions of sets ordered by refinement, which we denote as G_i.

We use combinatorial and algebraic methods to find the ndepths for small posets in G_i. We show that for posets of increasing size in G_i, new depth is strictly non-decreasing, and furthermore we show that $ndepth[G_i] \geq [8i/29]$ for all i. We also find that for all i, $ndepth[G_i] \leq i$ through the proof that $ndepth[G_{i+1}] \leq ndepth[G_i] + 1$.

*MIT-PRIMES.

1 Introduction

What is now called the Stanley depth of a $\mathbb{Z}^n$ graded module over a commutative ring S was first defined by R. P. Stanley [1] in 1982 in application to the partitioning of simplicial complexes. Stanley conjectured that the Stanley depth of such a module was always at least the depth of the module. There exists no general algorithm for computing Stanley depth, however in 2009 Herzog, Vladoiu, and Zheng [2] established a connection between the Stanley depths of a certain type of module, the monomial ideal, and an analogous property of related posets (partially ordered sets). Posets themselves are combinatorial objects with ties and applications to such varied fields of mathematics as topology, commutative algebra, representation theory, and group theory. The connection between the Stanley depth of monomial ideals and this new depth of posets provides an important link between combinatorics and commutative algebra. It naturally raises the question of what the new depth, which we will call ndepth, is for certain well-known classes of posets. The study of ndepth not only has the potential to answer questions in commutative algebra, but also may reveal previously unknown combinatorial properties of various classes of posets. In 2009, Biro, Howard, Keller, Trotter and Young [3] found the new depth for the poset of nonempty subsets of $\{1, 2, ..., n\}$ ordered by inclusion to be at least $\lceil n/2 \rceil$ and were able to use this result to prove that the Stanley depth of a monomial ideal $(x_1, \ldots x_n)$ is greater than or equal to $\lceil n/2 \rceil$. In 2010, Wang [4] showed that the ndepth of the product of chains $n^k \setminus 0$ is $(n-1)\lceil k/2 \rceil$. We now investigate ndepth for the class, which we will call G_i, of posets of nonempty partitions of a set of size i ordered by refinement. Through combinatorial and algebraic approaches, we seek to understand properties of ndepth for this class of posets and ultimately to find values or bounds for $ndepth[G_i]$ for all i.

We will begin with Section 2, in which we will define terms related to ndepth and posets, as well as other terms that will be used throughout the paper. Then, in Section 3, we will present our findings. Subsection 3.1 will show the values of $ndepth[G_i]$ for certain i. Subsection 3.2 will consist of the proofs of certain properties of the sequence $ndepth[G_i]$ as i increases. The lemmas that will be proven in Subsections 3.1 and 3.2 will be used in Subsection 3.3 to prove upper and lower bounds for $ndepth[G_i]$, first for certain values of i, and then for all i. These bounds are described by the theorems below.

Theorem 1. *For all i, $ndepth[G_i] \leq i - 1$.*

Theorem 2. *If $ndepth[G_n] = k$, then $ndepth[G_{3n}] \geq \min(k + n - 1, 3k, \lceil 3(n-1)/2 \rceil)$.*

Theorem 3. *For all i, $\lceil 8/29 \rceil \leq ndepth[G_i]$.*

Theorem 3 relies on lower bounds on $ndepth[G_i]$ for certain i established through the use of Theorem 2. The proofs of these theorems involve the calculation of $ndepth[G_i]$ for small i as well as the proof that $ndepth[G_i]$ is nondecreasing in i. We will also establish some limits on the speed at which $ndepth[G_i]$ grows as i increases, on which Theorem 1 is based.

We will then conclude in Section 4 with a discussion of possible directions of further research into this topic.

2 Definitions, Notation, and Methods

2.1 Posets

A poset (partially ordered set) is a well-known combinatorial object. We denote the relation between comparable elements of posets as $\leq$. In addition, we use *Hasse diagrams* to graphically represent posets. Below, we define concepts related to posets that we will use in this paper.

Definition 1. *For any poset, an* interval $I = [X_a, X_b]$ *is defined to include all elements X_c such that $X_a \leq X_c \leq X_b$.*

Definition 2. *An* interval partition *P of a poset G is a partition of G into non-empty intervals such that each element of G is in exactly one interval.*

Definition 3. *The* rank *of an element X of G, $\rho[X]$, is defined inductively such that the minimum value of $\rho[X]$ in every poset is 0, and $\rho[X_1] = \rho[X_2] + 1$ if $X_1 > X_2$ and there are no elements X_3 such that $X_1 > X_3 > X_2$.*

Definition 4. *The* depth *of an element of G, depth[X], is the maximal length of a chain $X > X_1 > X_2 > \ldots > X_n$ in G.*

Because G is strongly ordered, it is always true that $\rho[X]$ is one less than the length of the longest chain $X > X_1 > X_2 > \ldots > X_n$ in G; therefore, $\rho[X] = depth[X] - 1$.

Definition 5. *We define a* level *L of a poset to include all elements of depth L.*

Definition 6. *A* product *of two posets A and B is a new poset $A \times B$ such that for any pair of elements x in A and y in B there exists a corresponding element (x, y) in $A \times B$, which satisfies the property that $(x, y) \leq (x', y')$ in $A \times B$ if and only if $x \leq x'$ in A and $y \leq y'$ in B.*

Definition 7. *A* product of intervals *$I_A \times I_B$, for $I_A = [x, y]$ and $I_B = [x', y']$, is defined as the interval $[(x \times x'), (y \times y')]$.*

2.1.1 Refinement

A well-known class of posets is the class of posets of partitions of sets ordered by refinement.

Definition 8. *We call the partition of all elements of a set S into non-overlapping subsets a* set-partition.

Definition 9. *We define the* refinement *ordering to order set-partitions such that a set-partition X_a is finer than X_b if all subsets of X_a are within some subset in X_b.*

We will say if X_a is finer than X_b, then X_b is coarser than X_a and $X_a < X_b$.

Definition 10. *We define a specific class of posets G_i to include the posets which contain, as a set, all the non-empty set-partitions of a set with i elements, ordered by refinement.*

2.2 Ndepth

In [2], Herzog et. al. defined an analog of Stanley depth applicable to posets. Wang [4] called this new depth *ndepth*. Ndepth is defined below as it relates to intervals, interval partitions, and posets.

Definition 11. *The* ndepth of an interval I = *[X_a, X_b], denoted as* $ndepth[I]$, *is* $depth[X_b]$.

Definition 12. *The* ndepth of an interval partition, $ndepth[P]$, *is defined as:*

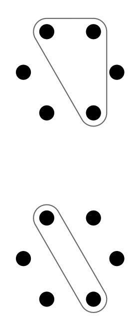
$$ndepth[P] := \min_{[X_a, X_b] \in P} depth[X_b]\,.$$

Definition 13. *The* ndepth of a poset, *ndepth[G], is defined as:*

$$ndepth[G] := \max_{[P]} ndepth[P].$$

2.3 Rotations

Sets can be represented as a series of points around a circle. Each set-partition may be represented graphically as a grouping of these points, as shown in Figure 1.



(X_b)

(X_a)

Figure 1: Two set-partitions, X_a and X_b, with X_b coarser than X_a.

Definition 14. *We define a* rotation *of a set-partition, $rot_n[X]$, to be a set-partition obtainable by turning the original partition n times clockwise around the circle.*

Definition 15. *We define the nth rotation of an* interval*, $I = [X_a, X_b]$, as $[rot_n[X_a], rot_n[X_b]]$.*

2.4 Classes

We introduce the idea of *classes* to further categorize set-partitions. Each element in G_i can be said to belong to exactly one class, determined by the sizes of the subsets in the partition, as shown in Figure 2.

Definition 16. *We define a* class $(n_1, n_2, \ldots, n_k)$*, where $n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_k$, to include all set-partitions which have subsets of size $n_1, \ldots, n_k$.*

Figure 2: Two set-partitions in class (3,1,1,1). If the left set-partition is X, then the right set-partition is $rot_3[X]$, the rotation of X 3 times clockwise around the circle.

3 Results

3.1 Values of $ndepth[G_i]$

For values of i up to 4, there are simple solutions for the value of $ndepth[G_i]$, shown in Table 1.

i	1	2	3	4
$ndepth[G_i]$	-	1	1	2

Table 1: Values of $ndepth[G_i]$ for i up to 4

For values of i greater than 4, it is useful to establish a relationship between the properties of a set-partition and the depth of the corresponding element in G_i.

Lemma 1. *The depth of a set-partition with n subsets in class $(s_1, s_2, \ldots, s_n)$ is* $\sum_{i=1}^{n}(s_i - 1)$.

Proof. Within the poset of all partitions of a set of size i, the rank of a partition of a set of size i into n subsets is $i - n$, and so the depth of such a partition is $i - n + 1$. Since the posets G_i are identical to these posets with the removal of the bottom level, the depth of the partition of the set of size i into n subsets within G_i is $i - n$. Because the sum of the numbers of points in all the subsets adds to the total number of points in the set, $i - n = (\sum_{i=1}^{n} s_i) - n = \sum_{i=1}^{n}(s_i - 1)$. □

Lemma 2. *We have that* $ndepth[G_5] = 3$.

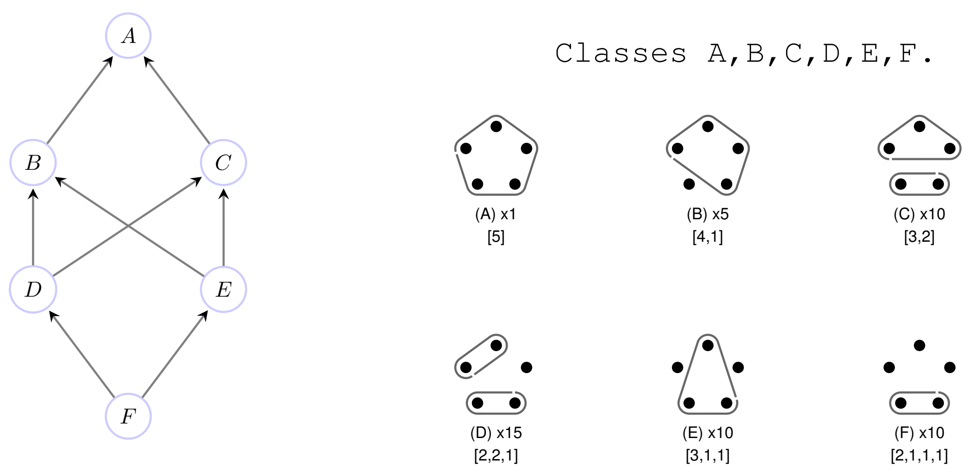


Figure 3: Hasse diagram of the relationship between classes of elements in G_5.

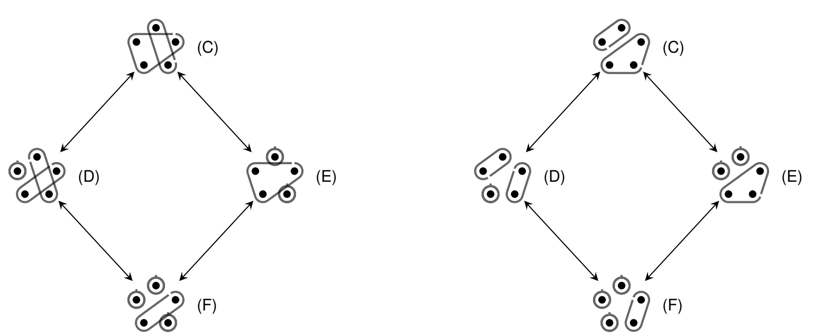


Figure 4: Structure of intervals of the form $[(F), (C)]$, as named in Figure 3.

Proof. Figure 3 represents the relationship between the 6 classes of elements in G_5 as a Hasse diagram.

The ndepth of the poset is n if and only if n is the maximal level for which it is possible to partition the set so that no interval has a maximal element with level $l \leq n$.

A possible partitioning of G_5 such that no interval has a maximal element with level $l \leq 3$ is as follows:

Let all elements of the form (F) be in an interval of the form $[(F),(C)]$ such that the intervals shown in Figure 4 and all rotations of each of the intervals shown in Figure 4 are intervals in the partition of G_i. Also, include all elements of form (D) which are not already within one of the previously described intervals of the form $[(F),(C)]$ in an interval of form $[(D),(B)]$. All elements of the form (D) which are not already within an interval of form $[(F),(C)]$ are rotations of one another, and all elements of form (B) are also rotations of each other, so 5 intervals $[(D),(B)]$ may be made containing all remaining elements of form (D) which are non-overlapping rotations of each other. These intervals are non-overlapping. Each rotation of an interval overlaps with no other rotation of the interval, and because the two intervals in Figure 4 are not rotations of each other, none of their rotations overlap.

The only remaining element in the poset is element A, which may be placed in an interval by itself to complete an interval partition of ndepth 3. Thus, $ndepth[G_5] \geq 3$.

It is impossible to create an interval partition of ndepth 4. There is only one element at level 4, so there can be at most one interval of depth greater than 3, but G_5 cannot be partitioned into only a single interval. Therefore, $ndepth[G_5]$ is 3, as shown in Table 2. $\square$

i	1	2	3	4	5
$ndepth[G_i]$	-	1	1	2	3

Table 2: Known values of $ndepth[G_i]$

3.2 Properties of $ndepth[G_i]$

Lemma 3. *The sequence $ndepth[G_i]$ is non-decreasing in i.*

Proof. Suppose for some i that $ndepth[G_i] = l$. It is sufficient to show that $ndepth[G_{i+1}] \geq l$. There exists some partition L of G_i such that $ndepth[L] = l$. We will use the partition L to construct a partition L' of G_{i+1} such that $ndepth[L'] = l$. This will imply the result.

Recall that G_{i+1} represents partitions of a set of $i+1$ points. We pick one of these points to be a "special point". We say that a partition g in G_{i+1} "contains" the special point if the special point is in a subset containing 2 or more points.

Consider all the elements of G_{i+1} which do not contain the special point. These elements form a subposet of G_{i+1} which is isomorphic to G_i. Therefore, we can use the partition L to partition this subposet.

What remains is to partition the elements that contain the special point. For all $n \leq i+1$, let all set-partitions which contain only one subset of size n which includes the special point be elements at the bottom of an interval. Let the top of each interval be the element containing that subset of size n as well as one subset containing all the rest of the points.

We claim that these intervals partition the elements that contain the special point and have depth less than l. All such elements are in some interval, since each contains the special point in some subset of size n, with some partition of the other points that is coarser than the partition of all the other points into a subset but finer than the empty partition of those points. Furthermore no two such intervals overlap, since each interval contains the special point in a unique subset of points, and no two elements in different intervals share this subset.

Let L' be the partition obtained by partitioning G_{i+1} into the intervals described above. The value $ndepth[L']$ is equal to the minimum of the ndepth of all intervals and of all sub-partitions in L'. The ndepth of the partition of the elements of G_{i+1} which do not contain the special point l, since the partition of those elements is isomorphic to L. The ndepth of the partition of the elements which do contain the special point is $i-1$, since all the top elements of the intervals within the partition contain two subsets, and all elements which are set partitions into two subsets are in level $i-1$. Since $ndepth[L']$ is the minimum of the ndepth of its sub-partitions, $ndepth[L'] = \min[l, i-1]$. For all i, $l \leq i-1$, since $l = ndepth[G_i]$ and $max[ndepth[G_i]] = i-1$. Thus, $ndepth[L'] = l$, and if $ndepth[G_i] = l$, then $ndepth[G_{i+1}] \geq l$.

□

Lemma 4. *If* $ndepth[G_i] = l-1$, *then* $ndepth[G_{i+1}] \leq l$.

Proof. It suffices to show that if G_{i+1} has ndepth l, then $ndepth[G_i] \geq l-1$.

There is a partition P of G_{i+1} with depth l. The set of $i+1$ points contains the set of i points as well as a special point. Partition P can include 3 types of intervals: those in which neither the bottom nor the top element contains the special point, those in which the top element but not the bottom element contains the special point, and those in which the bottom and top elements both contain the special point. Construct a partition P' of G_i with depth $l-1$ as follows.

The intervals in P which do not include the special point may be viewed as intervals in P' by removing the special point from each element within the interval. The ndepth of each of these new intervals is at least l.

The intervals in P which contain the special point in the top element but not the bottom element contain a set-partition on the level below the top element which resembles the top element, except in that it does not contain the special point. This element has depth of at least $l-1$. In the intervals in P, all the elements which do not contain the special point are within the interval between this element and the bottom element. If the special point is removed from each element in these intervals, they can be seen as intervals in P'. The ndepths of each of these intervals is at least $l-1$.

The two kinds of intervals described above do not overlap, because the intervals which generate them do not overlap in P. In addition, all elements in G_i must be contained within these intervals, since all elements in G_{i+1} not containing the special point are in one of the intervals in P from which these intervals were derived. Therefore these intervals partition G_i completely.

Partition P' of G_i has depth $l-1$. Thus if $ndepth[G_i] = l-1$, then $ndepth[G_{i+1}] \leq l$.

□

3.3 Bounds for $ndepth[G_i]$

Theorem 1. *For all $i \geq 2$, $ndepth[G_i] \leq i-1$.*

Proof. We know from Lemma 4 that $ndepth[G_{i+1}] \leq ndepth[G_i] + 1$. Therefore, the greatest possible difference $ndepth[G_{i_a}] - ndepth[G_{i_b}]$ between two values of i, i_b and i_a, is $i_a - i_b$. Since we have the initial case that $ndepth[G_2] = 1$, $ndepth[G_i] \leq i-1$ for all $i \geq 2$. □

Theorem 2. *If $ndepth[G_n] = k$, then $ndepth[G_{3n}] \geq \min(k+n-1, 3k, \lceil 3(n-1)/2 \rceil)$.*

Proof. There is a partition P of G_n which has ndepth k. We will construct a partition P' of G_{3n} such that $ndepth[P'] \geq \min(k+n-1, 3k, \lceil 3(n-1)/2 \rceil)$. Let posets G_A, G_B, and G_C be isomorphic to G_n through particular isomorphisms f_A, f_B, and f_C, respectively, and let *clusters* A, B, and C be sets of n points, the partitions of which are ordered in G_A, G_B, and G_C, respectively. Let G_{3n} be the product $G_A \times G_B \times G_C$.

Call a subset *varied* if it contains elements in two or three different clusters, and *non-varied* if it contains only elements within a single cluster. There exist three types of set-partitions: those containing only non-varied subsets within one or two of the three clusters, those containing only non-varied subsets within all three clusters, and those which contain varied subsets.

For every interval I in P, construct an interval I' in P' as follows: Taking the bottom element in I, we choose the corresponding element in A by using f_A. By the inclusion map this specifies an element in G_{3n}, which is the bottom element of I'. Taking the top element in I, we choose the corresponding element in A by using f_A. We add the set of all points in cluster B to get a set partition of G_{3n}, which is the top element of I'.

The intervals in P' which can be formed the same way I' was formed compose a class c_1 of intervals which have a bottom element containing only subsets in cluster A, and a top element containing cluster B as a subset as well as non-varied subsets in cluster A. Because $A \cong B \cong C$, we can use a method similar to the one described above to construct a class of intervals, c_2, with bottom elements containing only subsets in cluster B and top elements containing a subset including all of cluster C and subsets in cluster B, through f_B. Similarly, we can construct a class of intervals, c_3, with bottom elements containing only subsets in cluster C and top elements containing a subset including all of cluster A and subsets in cluster C, through f_C.

For every interval I in P, use f_A to create an interval with the corresponding elements in G_A. Say that all possible intervals created in this way comprise a partition P'_A of A. Then for every I use f_B to create an interval with the corresponding elements in G_B, and let these intervals be in a partition P'_B of B. Finally for every I use f_C to create an interval with the corresponding elements in G_C which is in a parition, P'_C, of C. We can then create a class c_4 of intervals which are the products of all possible combinations of one interval in P'_A, one interval in P'_B, and one interval in P'_C.

Finally, we create a class c_5 of all intervals which have a bottom element including only varied subsets, and a top element including those same subsets as well as a non-varied subset containing all the points in A not already contained in a subset, a non-varied subset containing all the points in B not already contained in a subset, and a non-varied subset containing all the points in C not already contained in a subset.

All set-partitions which contain only subsets which are non-varied and entirely within one or two clusters are included within some interval described in c_1, c_2, or c_3. The intervals in c_1 include all set-partitions with non-varied subsets in both A and B or in A alone, the intervals in c_2 include all set-partitions with non-varied subsets in both B and C or in B alone, and the intervals in c_3 include all set-partitions with non-varied subsets in both C and A or in C alone.

All set-partitions containing non-varied subsets within each of the three clusters are within some interval in c_4. This is true because A, B, and C are each completely partitioned by P'_A, P'_B, and P'_C, respectively, and the products of the intervals in the partitions will include all possible combinations of subsets in each of the partitions, which include all non-empty non-varied subsets.

All set-partitions with some positive number of varied subsets will be in an interval with a bottom element consisting of only those varied subsets and a top element with those varied subsets as well as subsets containing all of the rest of the points in each cluster. Thus, all set-partitions in G_{3n} containing a varied subset are within some interval in c_5.

All set-partitions in G_{3n} are therefore included in some interval in the 5 classes.

The intervals in each of these classes do not overlap with intervals in other classes. All intervals in c_5 contain only set-partitions including varied subsets. These do not occur in any of the other classes of intervals; thus, the intervals in c_5 do not overlap with the intervals in the other classes. All intervals in c_4 contain only set-partitions with only non-varied subsets in all three clusters, which also do not occur in any other class. Finally, the intervals in each of c_1, c_2, and c_3 do not overlap with intervals in the other two classes because the set-partitions in the intervals in each class include non-empty subsets in a unique combination of clusters. Intervals in c_1 contain only set-partitions with subsets in A or A and B; intervals in c_2 contain only set-partitions with subsets in B or B and C; intervals in c_3 have only set-partitions with subsets in C or C and A.

The intervals within each class also do not overlap amongst themselves. A set-partition in c_1 is within an interval which may be uniquely found through the partition of the points in A within the set-partition. The partition of A within the set-partition can be mapped to an element in G through f_A^{-1}, and this element in G is within a single interval in G. Since there is a one-to-one correlation between intervals in G and intervals in class c_1 through the method used to generate c_1, it must be true that each element in A, which maps to an element in G, which is in exactly one interval in G, must be within exactly one interval in the class c_1. The same is true for c_2 and c_3 by a similar argument.

No two intervals in c_4 share a set-partition, because the interval in which a set-partition in c_4 is can be uniquely determined as a product of three intervals. For each of A, B, and C, select the set of subsets of the set-partition which exist within that cluster, and take the product of the three intervals, in P'_A, P'_B, and P'_C, respectively, to which these sets of subsets belong.

Finally, no two intervals in c_5 contain the same set-partition, because the interval in which a set-partition in c_5 is can be uniquely determined to be the interval with a bottom element including only the varied subsets which exist within that set-partition.

Thus, we can construct a partition P' including all of the intervals in all 5 classes described above, which will completely partition G_{3n}.

The ndepths of intervals in c_1, c_2, and c_3 are at least $k + n - 1$. Intervals in c_1 have a top element with subsets in A which have a total depth of at least k, as well as a subset containing all points in B which has a depth of $n - 1$, for a total depth of at least $k + n - 1$. By similarity, intervals in c_2 and c_3 also have a top element with depth of at least $k + n - 1$.

The ndepths of intervals in c_4 are at least $3k$, since the top element of each of the intervals includes some set of subsets in each of A, B, and C, each with a total depth of at least k, for a total depth of at least $3k$.

The ndepths of intervals in c_5 are at least $\lceil 3(n-1)/2 \rceil$, because there are $3n$ points in total in the set, and each top element of an interval in c_5 is a complete partition of these points, with a minimum ndepth of $3n/2$, unless the bottom element leaves only one point in a cluster uncontained. In that case, it is possible to have up to three uncontained points in the top element, one for each cluster, and the minimum possible ndepth for these top elements is $\lceil 3(n-1)/2 \rceil$, which is then also the minimum ndepth of the intervals in c_5.

The ndepth of P' is therefore at least $\min(k+n-1, 3k, \lceil 3(n-1)/2 \rceil)$. Since $ndepth[G_{3n}] \geq ndepth[P']$, we must have $ndepth[G_{3n}] \geq \min(k+n-1, 3k, \lceil 3(n-1)/2 \rceil)$.

□

Theorem 3. *For all i, $ndepth[G_i] \geq 8i/29$.*

Proof. From Theorem 2, we know that in the sequence $ndepth[G_i]$, $ndepth[G_{3i}] \geq \min(ndepth[G_i] + i - 1, 3 \cdot ndepth[G_i], \lceil 3(i-1)/2 \rceil)$ for all i. Since $\min(ndepth[G_i] + i - 1, 3 \cdot ndepth[G_i], \lceil 3(i) - 1)/2 \rceil)$ increases as $ndepth[G_i]$ increases, we know that if $ndepth[G_i] \geq d$ for some integer d and some i, then $ndepth[G_{3i}] \geq \min(d + i - 1, 3d, \lceil 3(i-1)/2 \rceil)$. We will now show that $ndepth[G_i] \geq 8i/29$ for all i, using Theorem 2, Lemma 3, and Lemma 4.

In order to show this linear lower bound, we first establish lower bounds for $ndepth[G_i]$ for certain i. If we have an integer d such that $ndepth[G_i] \geq d$ and $d \leq (i-1)/2$, then since $ndepth[G_{3i}] \geq \min(d+i-1, 3d, \lceil 3(i-1)/2 \rceil)$ and $\min(d+i-1, 3d, \lceil 3(i-1)/2 \rceil) = 3d$ in this case, we have that $ndepth[G_{3i}] \geq 3d$. Repeating this argument, we can show that $ndepth[G_{9i}] \geq 9d$. Continuing by induction, we see that for all d and i such that $ndepth[G_i] \geq d$ and $d \leq (i-1)/2$, and for all integer n, we have that $3^n \cdot d \leq ndepth[G_{i \cdot 3n}]$.

We know from Table 2 that $ndepth[G_4] = 2$, $ndepth[G_5] = 3$, and $ndepth[G_6] \geq 3$. Therefore, we know that $ndepth[G_{12}] \geq 5$, $ndepth[G_{15}] \geq 6$, and $ndepth[G_{18}] \geq 8$. Based on these values, we know that $ndepth[G_{12 \cdot 3^n}] \geq 5 \cdot 3^n$, $ndepth[G_{15 \cdot 3^n}] \geq 6 \cdot 3^n$, and $ndepth[G_{18 \cdot 3^n}] \geq 8 \cdot 3^n$. These values are shown in Figure 5 as points on, respectively, the green line, $y = 5x/12$; the orange line, $y = 2x/5$; and the red line, $y = 4x/9$. We will call these points *bounded points*.

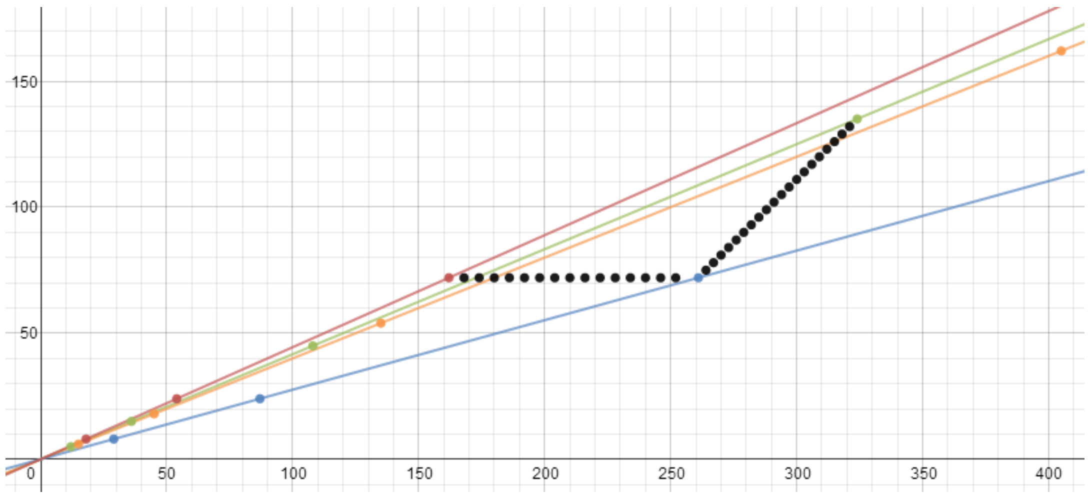


Figure 5: For n an integer, lower bounds for $ndepth[G_i]$ for: $i = 3^n \cdot 18$, in red; $i = 3^n \cdot 15$, in orange; $i = 3^n \cdot 12$, in green; and all i, in blue. The dotted lines represent restrictions on values of i between the colored points based on the fact that $ndepth[G_i]$ is non-decreasing and the fact that $ndepth[G_{i+1}] \leq ndepth[G_i] + 1$.

Because $ndepth[G_i]$ is non-decreasing and $ndepth[G_{i+1}] \leq ndepth[G_i] + 1$, we can place bounds on the values $ndepth[G_i]$ for intermittent values of i. In general, the lowest ratios of $ndepth[G_i]/i$ can be found at the intersections of the lines representing these restrictions. Consider, for n an integer, any bounded point $(18 \cdot 3^n, 8 \cdot 3^n)$, on the red line in Figure 5, and the next bounded point, which is at $(12 \cdot 3^{n+1}, 5 \cdot 3^{n+1})$ and on the green line. Because $ndepth[G_i]$ is non-decreasing, we know that for all $i \geq 18 \cdot 3^n$, it must be true that $ndepth[G_i] \geq 8 \cdot 3^n$. All points satisfying this are above the line $ndepth[G_i] \geq 8 \cdot 3^n$, which is shown as a horizontal dotted line in Figure 5. Similarly, because $ndepth[G_{i+1}] \leq ndepth[G_i] + 1$ for all i, we must have $ndepth[G_i] \geq i - 7 \cdot 3^{n+1}$. All points satisfying this are above the line $ndepth[G_i] \geq i - 7 \cdot 3^{n+1}$, which is shown as a dotted line with slope 1 in Figure 5. The intersection of these lines occurs at $(29 \cdot 3^n, 8 \cdot 3^n)$, and thus the lowest possible ratio of $ndepth[G_i]$ to i between these two points is $8/29$. Using a similar method, we find that the lowest possible ratio of $ndepth[G_i]$ to i between a bounded point on the green line and the next point on the orange line is $5/14$, and the lowest possible ratio of $ndepth[G_i]$ to i between a bounded point on the orange line and the next point on the red line is $3/8$. Since $3/8 > 5/14 > 8/29$, we can say that for all i, $ndepth[G_i] \geq 8/29$. This linear lower bound is shown in Figure 5 as the blue line.

□

4 Discussion, Conclusion and Further Work

In this paper we have applied the concept of ndepth to posets, G_i, of set-partitions ordered by refinement. We developed tools to better understand the properties of set-partitions and interval partitions, and employed these ideas to find $ndepth[G_i]$ for small values of i and to find bounds for $ndepth[G_i]$ for all i. These results are significant because they extend the idea of ndepth, developed in [2] and [3], to a class of posets which it had not formerly been applied to. In a broader sense, these results are also notable because they expand on the idea of Stanley depth, which is an important concept linking algebraic topology and combinatorics.

Due to the linear upper and lower bounds that we have obtained, we expect the sequence $ndepth[G_i]$ to be roughly linear. More specifically, based on the known values of $ndepth[G_i]$, we expect that $ndepth[G_i] \approx i/2$. Theorem 2 seems to further encourage this conclusion, since $k + n - 1$, $3k$, and $\lceil 3(n-1)/2 \rceil$ are most nearly equal when $k \approx n/2$, leading to a state where $\min(ndepth[G_i] + i - 1, 3 \cdot ndepth[G_i], \lceil 3(i-1)/2 \rceil)$ fluctuates between the three values.

Short of finding the sequence $ndepth[G_i]$ itself, work could be done to narrow the bounds on the values of $ndepth[G_i]$ for large i. In particular, we are interested in using the reverse of the process used to prove Theorem 2 to prove an upper bound for $ndepth[G_i]$. We would like to show that given a poset $G_{i \cdot n}$ with known ndepth k, there must be a partition of a smaller poset G_i with a ndepth of at least some value d. If so, this could show that if $ndepth[G_i] \leq d$, then

$ndepth[G_{n \cdot i}] \leq k$. Using the same method as in Theorem 3, it would be possible to use this result to find a lower upper bound for all i.

Other directions for future work include the calculation of $ndepth[G_i]$ for values $i \geq 6$, either with the methods we used in the proof of Lemma 2 or with computational software programs, and investigation of the properties of the partitions of G_i which yield the maximal $ndepth$.

5 Acknowledgments

I thank Prof. R. Stanley at MIT for suggesting this project. I also thank my mentor, Sergei Bernstein, for his invaluable help and guidance. I thank Dr. Tanya Khovanova for reviewing this paper and suggesting improvements. Finally, I thank the PRIMES program and all the PRIMES faculty for making this project possible.

References

[1] Stanley, R.P.: Linear Diophantine equations and local cohomology. Invent. Math. **68**(2), 175-193 (1982)

[2] Herzog, J., Vladoiu, M., Zheng, X.: How to compute the Stanley depth of a monomial ideal. J. Algebra (2009), doi:10.1016/j.jalgebra.2008.01.006, in press

[3] Biro, C., Howard, D., Trotter, M., Keller, W., Young, S.: *Interval Partitions and Stanley Depth*. Preprint (2008).

[4] Wang, Y: A New Depth Related to the Stanley Depth of some Power Sets of Multisets. arXiv:0908.3699v4.

[5] R. P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge Univ. Press, New York/Cambridge, UK, 1999.