Efficient Calculation of Determinants of Symbolic Matrices with Many Variables

Ziv Scully

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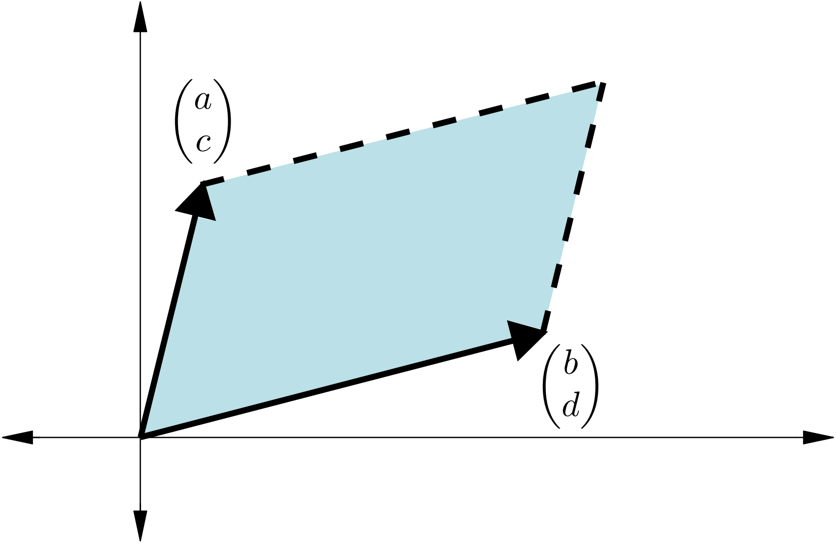
May 20, 2012

Vectors and Volumes



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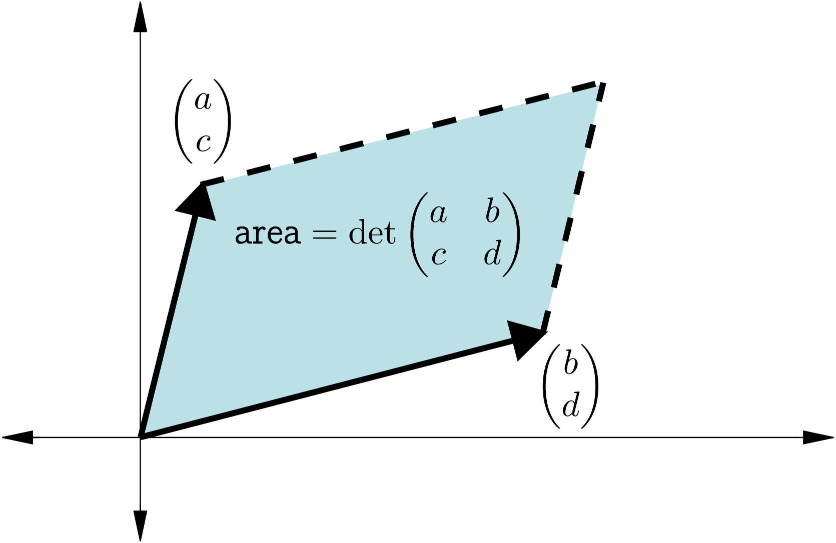

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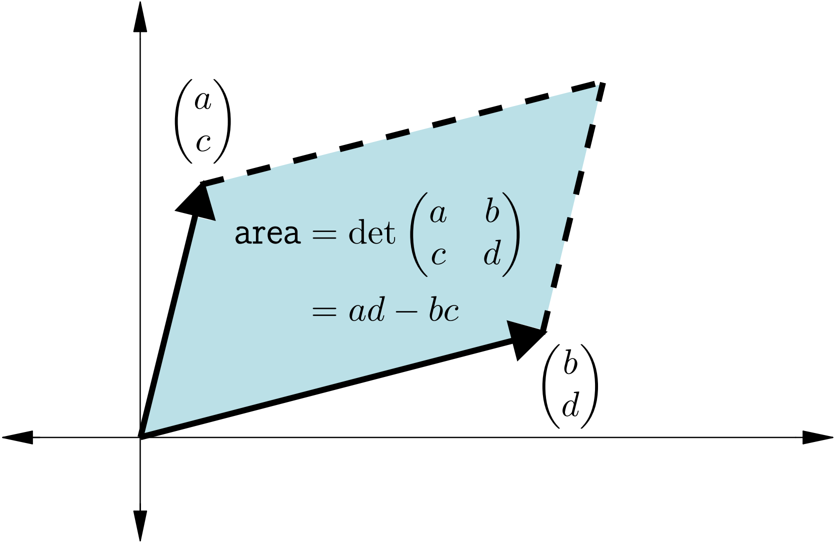

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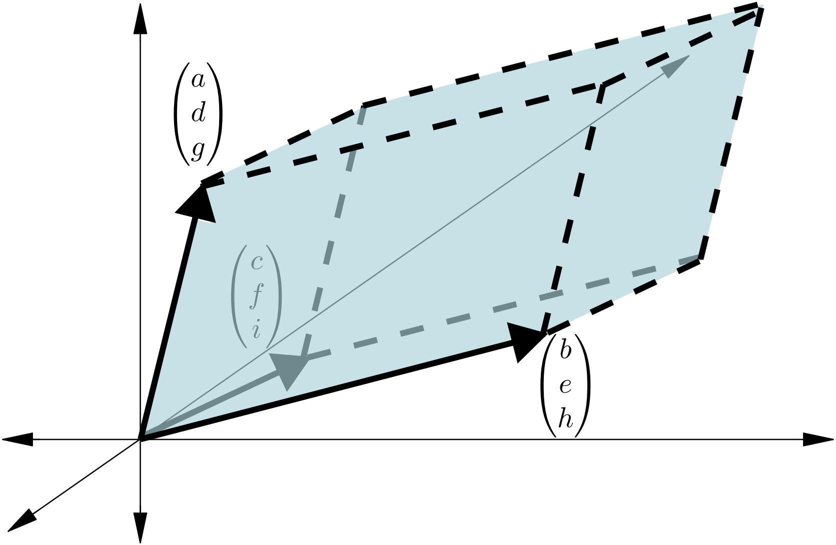

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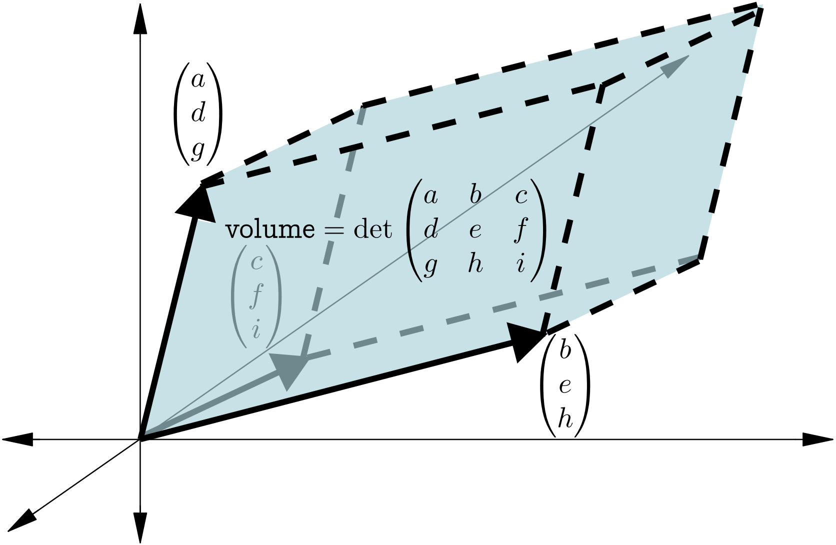

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$n \times n$ matrices:

$$\det(A) = \sum_{\sigma \in S_n} \left[\operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)} \right]$$

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We are interested in matrices with polynomial entries.

Minor Expansion

Naive calculation requires $\Theta(n!\,n)$ polynomial multiplications.

$$\det\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = \begin{matrix} afkp - aflo - agjp + agln + ahjo - ahkn \\ - bekp + belo + bgip - bglm - bhio + bhkm \\ + cejp - celn - cfip + cflm + chin - chjm \\ - dejo + dekn + dfio - dfkm - dgin + dgjm \end{matrix}$$

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Minor expansion requires $\displaystyle\sum_{i=2}^{n} i\binom{n}{i} \in \Theta(2^n n)$ polynomial multiplications.

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$$\det \begin{pmatrix} a & b & c & d \\ 0 & f & g & h \\ 0 & 0 & k & l \\ 0 & 0 & 0 & p \end{pmatrix} = afkp$$

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$$A_{i,j}^{(k+1)} = A_{i,j}^{(k)} - \frac{A_{i,k}^{(k)}}{A_{k,k}^{(k)}} A_{k,j}^{(k)}$$

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Fraction-free Gaussian elimination requires $\sum_{i=1}^{n} \Theta(i^2) \in \Theta(n^3)$ polynomial multiplications and divisions.

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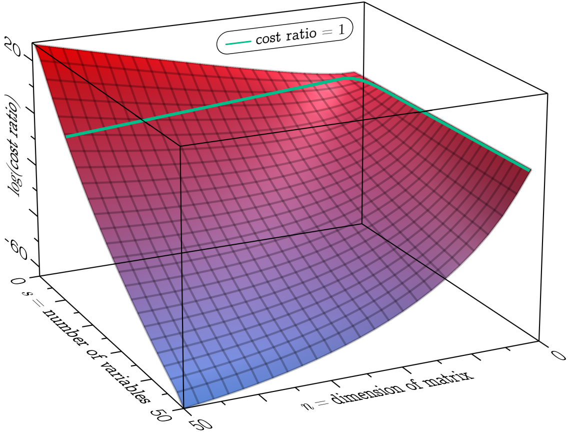
$$\text{cost ratio} = \frac{\text{cost of ME}}{\text{cost of FFGE}}$$



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- Absolute value of determinant is invariant under row swaps.

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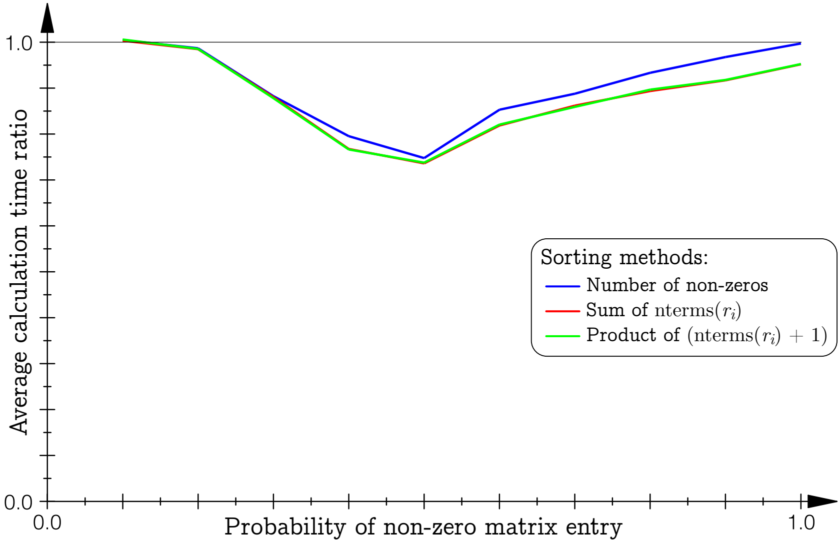
- Product of one more than number of terms for each entry of a row, $\prod_{i=1}^{n}(\mathrm{nterms}(r_i) + 1)$.

Data


1.0
0.0
Average calculation time ratio
0.0
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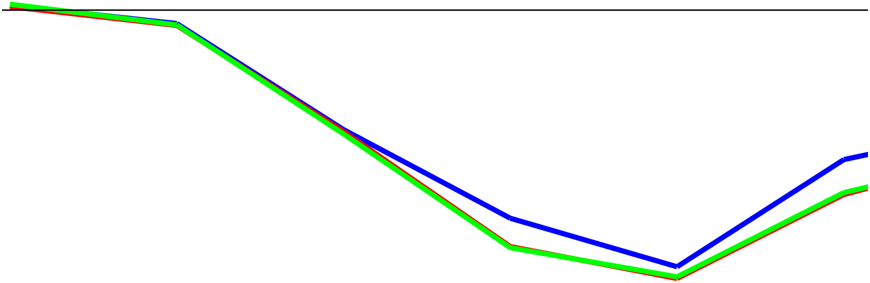
Probability of non-zero matrix entry
Sorting methods:
Number of non-zeros
Sum of nterms(r_i)
Product of (nterms(r_i) + 1)

Data



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- Machine learning.

Acknowledgments

- Dr. Tanya Khovanova, MIT, for mentoring me.
- Dr. Stefan Wehmeier, MathWorks, for suggesting the project and introducing me to the field.
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- MIT PRIMES, for giving me this unique research opportunity.
- The MathWorks, Inc., for providing software and supporting the research.