Modular representations of Cherednik algebras

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Second Annual MIT PRIMES Conference, May 19, 2012

Representations of Cherednik Algebras

- The Cherednik algebra $H_{\hbar,c}(G,\mathfrak{h})$ is a $\mathbb{Z}$-graded algebra
- We study the Cherednik algebra in positive characteristic
- The representations of the algebra we study are constructed from Verma modules $M_c(\tau)$ where τ is a representation of the group G
- $M_c(\tau)$ is equivalent to $\mathrm{Sym}(\mathfrak{h}^*)\otimes\tau$
- We construct a submodule $J_c(\tau)$ as the kernel of a bilinear form β_c which can be calculated with a computer: the lowest-weight representations of the Cherednik algebra are then $M_c(\tau)/J_c(\tau) = L_c(\tau)$
- The *Hilbert series* of L_c is $\displaystyle\sum_{i=0}^{\infty}(\dim(L_c)_i)t^i$.
- The main goal of the project is to be able to compute Hilbert series for all $L_c(\tau)$. We also study the free resolutions of some $L_c(\tau)$, allowing us to approximate certain modules with better-behaved ones

More Previous Results

- Latour, Katrina Evtimova, Emanuel Stoica, Martina Balagovic and Harrison Chen studied the Cherednik algebra for other groups
- Unlike them, we work with groups that are examples in char. 0 reduced mod p and higher rank
- We work with groups $G(m, r, n)$, which are n by n permutation matrices with entries that are m^{th} roots of unity such that the product of the entries is an $\frac{m}{r}^{th}$ root of unity
- With Carl Lian, we were able to find Hilbert series for the groups $G(1, 1, n)$ or S_n when $\hbar = 1$ for some special values of the parameter c for trivial τ: in general, we use generic c
- In the case when $G = S_n$, p divides n, τ is trivial, we were able to find Hilbert series for $L_c(\tau)$ and generators for $J_c(\tau)$ for $\hbar = 0$ and for $\hbar = 1, p = 2$
- For $G(m, m, 2)$ and $\hbar = 1$, we were able to find Hilbert series for $L_c(\tau)$ and generators for $J_c(\tau)$ for some τ

$\hbar = 0$, G(m,m,n) and G(m,1,n)

- The ideal J_c has behavior related to subspace arrangements in the case $G = G(m, 1, n)$, which includes the case $G = S_n$ $(m = 1)$
- Let X_i be the set of all $(x_1, \ldots, x_n)$ such that some $n - i$ of the coordinates are equal.
- Let $I_i^{(m)}$ be the ideal of X_i in degree m
- For $n \equiv i \pmod{p}$ with $0 \leq i \leq p - 1$ and $\hbar = 0$, the data suggests that J_c is generated by symmetric functions and $I_i^{(m)}$. L_c seems to be a complete intersection in X_i.
- For G(m,m,n) we see coordinate subspaces and the related ideals in the behavior of J_c
- We conjecture that when $n \equiv 0 \mod p$, the regular sequence is $x_1^m + \cdots x_n^m, x_1^{2m} + \cdots x_n^{2m}, \ldots, x_1^{(i-1)m} + \cdots x_n^{(i-1)m}$
- The exception is when $n \equiv 0 \mod p$, where J_c is generated by the squarefree monomials of degree p and the differences of the m^{th} powers of the x_i

Dihedral groups G(m,m,2), $\hbar = 0$

- Dihedral groups are the groups $G(m, m, 2)$, they can also be considered the group of symmetries of a regular m-gon
- Representations of the dihedral group take the form ρ_i for $0 \leq i < \frac{m}{2}$: these representations are equivalent to the standard 2-dimensional one, except roots of unity act by their i^{th} power (except for $i = 0$, which is the trivial representation)
- There are 1 or 3 additional representations based on tensoring the trivial representation by a character (for example, the sign representation), depending on the parity of m
- These are indexed by negative integers
- We use these representations as τ

- For $i \leq 0$, ρ_i has one basis vector e_1; for $i > 0$, ρ_i has two basis vectors e_1, e_2
- Let x_1 and x_2 be basis vectors of $\mathfrak{h}^*$
- The results in this case appear to be independent of characteristic
- If $i \leq 0$, then $x_1 * x_2 \otimes e_1, (x_1^m + x_2^m) \otimes e_1$ generate J_c
- If $i = 1$, then $x_1 \otimes e_1, x_1^3 \otimes e_2, x_2^3 \otimes e_1, x_2 \otimes e_2$ generate J_c
- If $1 < i < \frac{m}{2}$, then $x_1 \otimes e_1, x_1 \otimes e_2, x_2 \otimes e_1, x_2 \otimes e_2$ generate J_c unless m is even and $i = \frac{m}{2} - 1$
- If $i = \frac{m}{2} - 1$ and m is even, then $x_1 \otimes e_1, x_1^3 \otimes e_2, x_2^3 \otimes e_1, x_2 \otimes e_2$ generate J_c
- $m = 4$ is a special case since $1 = \frac{m}{2} - 1$

Dihedral group free resolutions

- Free resolutions can be calculated for $L_c(\rho_i)$ in most cases (let $A = \mathrm{Sym}(\mathfrak{h}^*)$)
- If $i \leq 0$, then the free resolution is:

$$0 \leftarrow L_c(\rho_i) \leftarrow \rho_i \otimes A \leftarrow \rho_i \otimes A(-2) \oplus \rho_i \otimes A(-m) \leftarrow \rho_i \otimes A(-m-2) \leftarrow 0$$

- If $i = 1$ the free resolution is:

$$0 \leftarrow L_c(\rho_1) \leftarrow \rho_1 \otimes A \leftarrow \rho_2 \otimes A(-1) \oplus \rho_2 \otimes A(-3) \leftarrow \rho_1 \otimes A(-4) \leftarrow 0$$

- If $1 < i < \frac{m}{2}$ (unless m is even and $i = \frac{m}{2} - 1$) the free resolution is:

$$0 \leftarrow L_c(\rho_i) \leftarrow \rho_i \otimes A \leftarrow \rho_i \otimes \mathfrak{h}^* \otimes A(-1) \leftarrow \rho_i \otimes \wedge^2 \mathfrak{h}^* \otimes A(-2) \leftarrow 0$$

- If $i = \frac{m}{2} - 1$, and m is even and greater than 8, the free resolution is:

$$0 \leftarrow L_c(\rho_i) \leftarrow \rho_i \otimes A \leftarrow (\rho_{-2} \oplus \rho_{-1}) \otimes A(-1) \oplus \rho_{\frac{m}{2}-4} \otimes A(-3) \leftarrow \rho_{\frac{m}{2}-3} \otimes A(-4) \leftarrow 0$$

The following transition matrix, for the case $G(5,5,2)$, expresses the characters of the $L_c(\tau)$ as alternating sums of the characters of the Verma modules $M_c(\tau)$, using the variable t to represent grading shifts:

$$\begin{pmatrix} (1-t^2)(1-t^5) & 0 & 0 & 0 \\ 0 & (1-t^2)(1-t^5) & 0 & 0 \\ 0 & 0 & 1+t^4 & -t \\ 0 & 0 & -t-t^3 & 1-t+t^2 \end{pmatrix}$$

The columns of this matrix represent $L_c(\tau)$ for the four representations of $G(5,5,2)$, while the rows represent $M_c(\tau)$ for the same four representations (in the order $\rho_{-1}, \rho_0, \rho_1, \rho_2$)

The inverse matrix shows the characters of the $M_c(\tau)$ in terms of the characters of the $L_c(\tau)$, with the fractional coefficient representing that the $L_c(\tau)$ are being infinitely summed. The baby Verma modules $M'_c(\tau)$ are equivalent to $M_c(\tau)$ quotiented by the invariants, which have degrees 2 and 5 for $G(5,5,2)$, so when we remove the fractional coefficient, the transitional matrix relates the baby Verma modules to the $L_c(\tau)$.
(Here the columns refer to the $M_c(\tau)$ and the rows to the $L_c(\tau)$, with the same indexing of representations.)

$$\frac{1}{(1-t^2)(1-t^5)}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+t^3 & t+t^2+t^3+t^4 \\ 0 & 0 & t+t^2 & 1+t+t^4+t^5 \end{pmatrix}$$

- Let x, y, z be basis vectors of $\mathfrak{h}^*$
- $G(m, m, 3)$ has one two-dimensional representation γ_0: it is equivalent to the standard three-dimensional representation with roots of unity acting trivially, quotiented by the sum of the variables, and it has two basis vectors e_1 and e_2
- The following results are true when $p > 2$
- In this case we conjecture that J_c is generated by $(x^m + y^m + z^m) \otimes e_1, (x^m + y^m + z^m) \otimes e_2, xyz \otimes e_1, xyz \otimes e_2, -x^m \otimes e_1 + z^m \otimes e_2, y^m \otimes e_1 + -x^m \otimes e_2$
- $G(m, m, 3)$ has $m - 1$ three-dimensional representations γ_i for $1 \leq i \leq m - 1$ equivalent to the standard three-dimensional representation, with roots of unity acting by their i^{th} power (three basis vectors e_1, e_2, e_3)
- In this case (unless $i = 1, p = 2$, or $m = 2$) we conjecture that J_c is generated by $x \otimes e_1, y \otimes e_2, z \otimes e_3, yz \otimes e_1, xz \otimes e_2, xy \otimes e_3, y^{m-i} \otimes e_1 + x^{m-i} \otimes e_2, z^{m-i} \otimes e_1 + x^{m-i} \otimes e_3, z^{m-i} \otimes e_2 + y^{m-i} \otimes e_3$

Further research

- We plan to find the expressions of the $M'_c(\tau)$ in terms of the $L_c(\tau)$ for the remaining cases for the dihedral group and the groups $G(m, m, 3)$ as well
- We also plan to find free resolutions for small cases of $G(m, r, n)$ and use K-theory in a similar way

Thanks!

Thanks to PRIMES for providing this opportunity, Pavel Etingof for suggesting this problem, and our mentor Steven Sam.