Schmidt Games and a Family of Anormal Numbers

Saarik Kalia and Michael Zanger-Tishler

Second Annual MIT PRIMES Conference

May 19, 2012

Schmidt Games

Two players, Alice and Bob, play a Schmidt game on a given set S, as follows:

Schmidt Games

Two players, Alice and Bob, play a Schmidt game on a given set S, as follows:

- Alice has a constant $0 < \alpha < 1$, and Bob has a constant $0 < \beta < 1$.

Schmidt Games

Two players, Alice and Bob, play a Schmidt game on a given set S, as follows:

- Alice has a constant $0 < \alpha < 1$, and Bob has a constant $0 < \beta < 1$.
- Bob begins by choosing an interval $B_1 \subset \mathbb{R}$.

Schmidt Games

Two players, Alice and Bob, play a Schmidt game on a given set S, as follows:

- Alice has a constant $0 < \alpha < 1$, and Bob has a constant $0 < \beta < 1$.
- Bob begins by choosing an interval $B_1 \subset \mathbb{R}$.
- Alice then chooses an interval $A_1 \subset B_1$, such that $|A_1| = \alpha |B_1|$.

Schmidt Games

Two players, Alice and Bob, play a Schmidt game on a given set S, as follows:

- Alice has a constant $0 < \alpha < 1$, and Bob has a constant $0 < \beta < 1$.
- Bob begins by choosing an interval $B_1 \subset \mathbb{R}$.
- Alice then chooses an interval $A_1 \subset B_1$, such that $|A_1| = \alpha |B_1|$.
- Bob then chooses an interval $B_2 \subset A_1$, such that $|B_2| = \beta |A_1|$.

Schmidt Games

Two players, Alice and Bob, play a Schmidt game on a given set S, as follows:

- Alice has a constant $0 < \alpha < 1$, and Bob has a constant $0 < \beta < 1$.
- Bob begins by choosing an interval $B_1 \subset \mathbb{R}$.
- Alice then chooses an interval $A_1 \subset B_1$, such that $|A_1| = \alpha |B_1|$.
- Bob then chooses an interval $B_2 \subset A_1$, such that $|B_2| = \beta |A_1|$.

Schmidt Games

- This process repeats infinitely. If the point of convergence $\bigcap_{k=0}^{\infty} B_k = \bigcap_{k=0}^{\infty} A_k$ is in S, Alice wins.

Schmidt Games

- This process repeats infinitely. If the point of convergence $\bigcap_{k=0}^{\infty} B_k = \bigcap_{k=0}^{\infty} A_k$ is in S, Alice wins.
- A set S is called (α, β)-winning, if for the given (α, β) and S, Alice can ensure victory, regardless of how Bob plays.

Schmidt Games

- This process repeats infinitely. If the point of convergence $\bigcap_{k=0}^{\infty} B_k = \bigcap_{k=0}^{\infty} A_k$ is in S, Alice wins.
- A set S is called (α, β)-winning, if for the given (α, β) and S, Alice can ensure victory, regardless of how Bob plays.
- For our purposes, if S is not (α, β)-winning, it is (α, β)-losing.

Schmidt Diagrams and Trivial Zones

We will explore the values of (α, β) for which a given set is winning. We therefore define the Schmidt Diagram of S, $D(S)$, as the set of all (α, β) for which S is winning.

Schmidt Diagrams and Trivial Zones

We will explore the values of (α, β) for which a given set is winning. We therefore define the Schmidt Diagram of S, $D(S)$, as the set of all (α, β) for which S is winning.

We can identify two "trivial zones" which are either winning or losing for every set by examing Alice or Bob's ability to center all of their intervals around a common point.

Schmidt Diagrams and Trivial Zones

We will explore the values of (α, β) for which a given set is winning. We therefore define the Schmidt Diagram of S, $D(S)$, as the set of all (α, β) for which S is winning.

We can identify two "trivial zones" which are either winning or losing for every set by examing Alice or Bob's ability to center all of their intervals around a common point.

Suppose that $S = \mathbb{R} \backslash \{x\}$. If $1 - 2\alpha + \alpha\beta < 0$, Bob can ensure victory as follows:

Schmidt Diagrams and Trivial Zones

We will explore the values of (α, β) for which a given set is winning. We therefore define the Schmidt Diagram of S, $D(S)$, as the set of all (α, β) for which S is winning.

We can identify two "trivial zones" which are either winning or losing for every set by examing Alice or Bob's ability to center all of their intervals around a common point.

Suppose that $S = \mathbb{R} \backslash \{x\}$. If $1 - 2\alpha + \alpha\beta < 0$, Bob can ensure victory as follows:

- Bob centers B_1 around x. Let $d = |B_1|$.

Schmidt Diagrams and Trivial Zones

We will explore the values of (α, β) for which a given set is winning. We therefore define the Schmidt Diagram of S, $D(S)$, as the set of all (α, β) for which S is winning.

We can identify two "trivial zones" which are either winning or losing for every set by examing Alice or Bob's ability to center all of their intervals around a common point.

Suppose that $S = \mathbb{R} \backslash \{x\}$. If $1 - 2\alpha + \alpha\beta < 0$, Bob can ensure victory as follows:

- Bob centers B_1 around x. Let $d = |B_1|$.
- The farthest right Alice's leftmost endpoint can be is at $x + \frac{d}{2} - d\alpha$.

Schmidt Diagrams and Trivial Zones

We will explore the values of (α, β) for which a given set is winning. We therefore define the Schmidt Diagram of S, $D(S)$, as the set of all (α, β) for which S is winning.

We can identify two "trivial zones" which are either winning or losing for every set by examing Alice or Bob's ability to center all of their intervals around a common point.

Suppose that $S = \mathbb{R} \setminus \{x\}$. If $1 - 2\alpha + \alpha\beta < 0$, Bob can ensure victory as follows:

- Bob centers B_1 around x. Let $d = |B_1|$.
- The farthest right Alice's leftmost endpoint can be is at $x + \frac{d}{2} - d\alpha$.
- The farthest left Bob's center can then be is $x + \frac{d}{2} - d\alpha + \frac{d\alpha\beta}{2}$. Bob obviously can then maintain the same center.

Trivial Zones

Using similar logic, we can reach the following conclusions about $D(S)$:

Trivial Zones

Using similar logic, we can reach the following conclusions about $D(S)$:

- S is dense $\iff D(S) \neq \emptyset \implies$ the region $\{(\alpha, \beta) : \beta \geq \frac{1}{2-\alpha}\}$ is winning.

Trivial Zones

Using similar logic, we can reach the following conclusions about $D(S)$:

- S is dense $\iff D(S) \neq \emptyset \implies$ the region $\{(\alpha, \beta) : \beta \geq \frac{1}{2-\alpha}\}$ is winning.
- S is a countable dense set $\implies D(S) = \{(\alpha, \beta) : \beta \geq \frac{1}{2-\alpha}\}$

Trivial Zones

Using similar logic, we can reach the following conclusions about $D(S)$:

- S is dense $\iff D(S) \neq \emptyset \implies$ the region $\{(\alpha, \beta) : \beta \geq \frac{1}{2-\alpha}\}$ is winning.
- S is a countable dense set $\implies D(S) = \{(\alpha, \beta) : \beta \geq \frac{1}{2-\alpha}\}$
- S is co-countable $\implies D(S)^C = \{(\alpha, \beta) : \beta \leq 2 - \frac{1}{\alpha}\}$

Trivial Zones

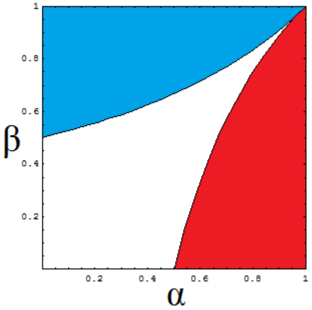
Using similar logic, we can reach the following conclusions about $D(S)$:

- S is dense $\iff D(S) \neq \emptyset \implies$ the region $\{(\alpha, \beta) : \beta \geq \frac{1}{2-\alpha}\}$ is winning.
- S is a countable dense set $\implies D(S) = \{(\alpha, \beta) : \beta \geq \frac{1}{2-\alpha}\}$
- S is co-countable $\implies D(S)^C = \{(\alpha, \beta) : \beta \leq 2 - \frac{1}{\alpha}\}$
- $S \neq \mathbb{R} \iff D(S)^C \neq \emptyset \implies$ the region $\{(\alpha, \beta) : \beta \leq 2 - \frac{1}{\alpha}\}$ is losing.

Trivial Zones

Using similar logic, we can reach the following conclusions about $D(S)$:

- S is dense $\iff D(S) \neq \emptyset \implies$ the region $\{(\alpha, \beta) : \beta \geq \frac{1}{2-\alpha}\}$ is winning.
- S is a countable dense set $\implies D(S) = \{(\alpha, \beta) : \beta \geq \frac{1}{2-\alpha}\}$
- S is co-countable $\implies D(S)^C = \{(\alpha, \beta) : \beta \leq 2 - \frac{1}{\alpha}\}$
- $S \neq \mathbb{R} \iff D(S)^C \neq \emptyset \implies$ the region $\{(\alpha, \beta) : \beta \leq 2 - \frac{1}{\alpha}\}$ is losing.



Lemmas

Here are a few lemmas for Schmidt Games:

Lemmas

Here are a few lemmas for Schmidt Games:

- If S is (α, β)-winning, it is also (α', β')-winning for $\alpha\beta = \alpha'\beta', \alpha > \alpha'$.

Lemmas

Here are a few lemmas for Schmidt Games:

- If S is (α, β)-winning, it is also (α', β')-winning for $\alpha\beta = \alpha'\beta', \alpha > \alpha'$.
- If S is (α, β)-winning, it is also $(\alpha(\alpha\beta)^n, \beta)$-winning for $n \in \mathbb{W}$.

Lemmas

Here are a few lemmas for Schmidt Games:

- If S is (α, β)-winning, it is also (α', β')-winning for $\alpha\beta = \alpha'\beta', \alpha > \alpha'$.
- If S is (α, β)-winning, it is also $(\alpha(\alpha\beta)^n, \beta)$-winning for $n \in \mathbb{W}$.
- If $S_1 \subset S_2$, then $D(S_1) \subseteq D(S_2)$.

Lemmas

Here are a few lemmas for Schmidt Games:

- If S is (α, β)-winning, it is also (α', β')-winning for $\alpha\beta = \alpha'\beta', \alpha > \alpha'$.
- If S is (α, β)-winning, it is also $(\alpha(\alpha\beta)^n, \beta)$-winning for $n \in \mathbb{W}$.
- If $S_1 \subset S_2$, then $D(S_1) \subseteq D(S_2)$.
- $D(S_1 \cap S_2) \subseteq D(S_1) \cap D(S_2)$ and $D(S_1 \cup S_2) \supseteq D(S_1) \cup D(S_2)$

Lemmas

Here are a few lemmas for Schmidt Games:

- If S is (α, β)-winning, it is also (α', β')-winning for $\alpha\beta = \alpha'\beta', \alpha > \alpha'$.
- If S is (α, β)-winning, it is also $(\alpha(\alpha\beta)^n, \beta)$-winning for $n \in \mathbb{W}$.
- If $S_1 \subset S_2$, then $D(S_1) \subseteq D(S_2)$.
- $D(S_1 \cap S_2) \subseteq D(S_1) \cap D(S_2)$ and $D(S_1 \cup S_2) \supseteq D(S_1) \cup D(S_2)$
- $D(S^C) \subseteq \sigma(D(S)^C)$, where S^C represents the complement of S and $\sigma(x, y) = (y, x)$.

Lemmas

Here are a few lemmas for Schmidt Games:

- If S is (α, β)-winning, it is also (α', β')-winning for $\alpha\beta = \alpha'\beta', \alpha > \alpha'$.
- If S is (α, β)-winning, it is also $(\alpha(\alpha\beta)^n, \beta)$-winning for $n \in \mathbb{W}$.
- If $S_1 \subset S_2$, then $D(S_1) \subseteq D(S_2)$.
- $D(S_1 \cap S_2) \subseteq D(S_1) \cap D(S_2)$ and $D(S_1 \cup S_2) \supseteq D(S_1) \cup D(S_2)$
- $D(S^C) \subseteq \sigma(D(S)^C)$, where S^C represents the complement of S and $\sigma(x, y) = (y, x)$.
- If F is a locally finite set, $S \neq \mathbb{R}$, and $S \cup F \neq \mathbb{R}$, then $D(S \cup F) = D(S) = D(S \backslash F)$.

Lemmas

Here are a few lemmas for Schmidt Games:

- If S is (α, β)-winning, it is also (α', β')-winning for $\alpha\beta = \alpha'\beta', \alpha > \alpha'$.
- If S is (α, β)-winning, it is also $(\alpha(\alpha\beta)^n, \beta)$-winning for $n \in \mathbb{W}$.
- If $S_1 \subset S_2$, then $D(S_1) \subseteq D(S_2)$.
- $D(S_1 \cap S_2) \subseteq D(S_1) \cap D(S_2)$ and $D(S_1 \cup S_2) \supseteq D(S_1) \cup D(S_2)$
- $D(S^C) \subseteq \sigma(D(S)^C)$, where S^C represents the complement of S and $\sigma(x, y) = (y, x)$.
- If F is a locally finite set, $S \neq \mathbb{R}$, and $S \cup F \neq \mathbb{R}$, then $D(S \cup F) = D(S) = D(S \backslash F)$.
- If $S' = kS + c$ is the set S scaled by a factor of k and shifted by c, then $D(S') = D(S)$.

Our Set

The set we will examine, denoted by $F_{b,w}$, is the set of numbers in whose base b expansion the digit w appears finitely many times.

Our Set

The set we will examine, denoted by $F_{b,w}$, is the set of numbers in whose base b expansion the digit w appears finitely many times.

For instance, the number $0.44444\overline{0123}_5$ would be in $F_{5,4}$.

Our Set

The set we will examine, denoted by $F_{b,w}$, is the set of numbers in whose base b expansion the digit w appears finitely many times.

For instance, the number $0.4444\overline{40123}_5$ would be in $F_{5,4}$.

If we define Z_k as the set of numbers which have a w as their k^{th} decimal place in their base b expansion, then we can define this set as the set of numbers which are in only a finite number of Z_k's.

Our Set

The set we will examine, denoted by $F_{b,w}$, is the set of numbers in whose base b expansion the digit w appears finitely many times.

For instance, the number $0.44444\overline{0123}_5$ would be in $F_{5,4}$.

If we define Z_k as the set of numbers which have a w as their k^{th} decimal place in their base b expansion, then we can define this set as the set of numbers which are in only a finite number of Z_k's.

Winning and Losing Zones

If Alice can disjoint her intervals from all the Z_k's beginning with a certain k, she can win. If the set of k for which Bob can contain his intervals in Z_k is unbounded, he can win. Therefore:

Winning and Losing Zones

If Alice can disjoint her intervals from all the Z_k's beginning with a certain k, she can win. If the set of k for which Bob can contain his intervals in Z_k is unbounded, he can win. Therefore:

- The region $\{(\alpha, \beta) : \alpha < \frac{1}{2}, \alpha\beta \geq \frac{1}{b}, \beta \geq \frac{1}{(b-1)(1-2\alpha)}\}$ is winning.

Winning and Losing Zones

If Alice can disjoint her intervals from all the Z_k's beginning with a certain k, she can win. If the set of k for which Bob can contain his intervals in Z_k is unbounded, he can win. Therefore:

- The region $\{(\alpha, \beta) : \alpha < \frac{1}{2}, \alpha\beta \geq \frac{1}{b}, \beta \geq \frac{1}{(b-1)(1-2\alpha)}\}$ is winning.
- The region $\{(\alpha, \beta) : \beta < \frac{1}{b+1} \vee (\beta = \frac{1}{b+1}, \log_b \alpha\beta \in \mathbb{Q})\}$ is losing.

Winning and Losing Zones

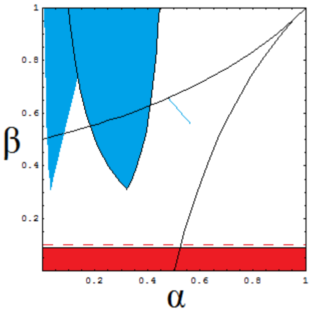
If Alice can disjoint her intervals from all the Z_k's beginning with a certain k, she can win. If the set of k for which Bob can contain his intervals in Z_k is unbounded, he can win. Therefore:

- The region $\{(\alpha, \beta) : \alpha < \frac{1}{2}, \alpha\beta \geq \frac{1}{b}, \beta \geq \frac{1}{(b-1)(1-2\alpha)}\}$ is winning.
- The region $\{(\alpha, \beta) : \beta < \frac{1}{b+1} \vee (\beta = \frac{1}{b+1}, \log_b \alpha\beta \in \mathbb{Q})\}$ is losing.
- Where b is the base

Winning and Losing Zones

If Alice can disjoint her intervals from all the Z_k's beginning with a certain k, she can win. If the set of k for which Bob can contain his intervals in Z_k is unbounded, he can win. Therefore:

- The region $\{(\alpha,\beta) : \alpha < \frac{1}{2}, \alpha\beta \geq \frac{1}{b}, \beta \geq \frac{1}{(b-1)(1-2\alpha)}\}$ is winning.
- The region $\{(\alpha,\beta) : \beta < \frac{1}{b+1} \vee (\beta = \frac{1}{b+1}, \log_b \alpha\beta \in \mathbb{Q})\}$ is losing.
- Where b is the base



Future Research

- We can extend the losing region to $\{(\alpha, \beta) : \beta < \frac{1}{b} \vee (\beta = \frac{1}{b}, \log_b \alpha\beta \in \mathbb{Q})\}$ for $F_{b,0}$ and $F_{b,b-1}$. Can we apply this extension to all other digits?

Future Research

- We can extend the losing region to $\{(\alpha, \beta) : \beta < \frac{1}{b} \vee (\beta = \frac{1}{b}, \log_b \alpha\beta \in \mathbb{Q})\}$ for $F_{b,0}$ and $F_{b,b-1}$. Can we apply this extension to all other digits?
- Can we show that the digit does not matter?

Future Research

- We can extend the losing region to $\{(\alpha, \beta) : \beta < \frac{1}{b} \vee (\beta = \frac{1}{b}, \log_b \alpha\beta \in \mathbb{Q})\}$ for $F_{b,0}$ and $F_{b,b-1}$. Can we apply this extension to all other digits?
- Can we show that the digit does not matter?
- Can we find a complete Schmidt Diagram for $F_{b,w}$?

Future Research

- We can extend the losing region to $\{(\alpha, \beta) : \beta < \frac{1}{b} \vee (\beta = \frac{1}{b}, \log_b \alpha\beta \in \mathbb{Q})\}$ for $F_{b,0}$ and $F_{b,b-1}$. Can we apply this extension to all other digits?
- Can we show that the digit does not matter?
- Can we find a complete Schmidt Diagram for $F_{b,w}$?
- Can we find a complete non-trivial Schmidt Diagram for any other set?

Acknowledgements

- Our mentor, Tue Ly, for assisting us in our research.

Acknowledgements

- Our mentor, Tue Ly, for assisting us in our research.
- Professor Dmitry Kleinbock for suggesting this project and providing feedback on our research so far.

Acknowledgements

- Our mentor, Tue Ly, for assisting us in our research.
- Professor Dmitry Kleinbock for suggesting this project and providing feedback on our research so far.
- Our parents for their continued support (and driving).

Acknowledgements

- Our mentor, Tue Ly, for assisting us in our research.
- Professor Dmitry Kleinbock for suggesting this project and providing feedback on our research so far.
- Our parents for their continued support (and driving).
- PRIMES for enabling us to work on this project.