Equivalence Classes of Permutations Created by Replacement Sets

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$\{123, 231\}$: A simple example

$\{123, 231\}$

$$2\underline{134}$$

$$\downarrow$$

$$2\underline{341}$$

$\{123, 231\}$: A simple example

$\{123, 231\}$

$$2134$$

$$\downarrow$$

$$\underline{234}1$$

$\{123, 231\}$: A simple example

$\{123, 231\}$

$$2134$$
$$\downarrow$$
$$\underline{234}1$$
$$\downarrow$$
$$\underline{342}1$$

How do we make equivalence classes?

If c adjacent letters in a permutation in S_n have the same order as a pattern in the replacement set, then they can be rearranged to have the order of any other pattern in the replacement set.

Definition

*An **equivalence class** is the set of permutations reachable from some given permutation.*

- Example: Consider $\{12, 21\}$. There is only one class. In S_3, $123 \equiv 132 \equiv 312 \equiv 321 \equiv 231 \equiv 213$.

Non-trivial example for $n = 4$

Consider $\{123, 132, 231\}$.

1234	2134	3241	4231	3214	4213	4321	4312
2314	2143	3124	4123				
1324	2341	3142	4132				
1342	3421						
1243	2431						
3412							
1432							
2413							
1423							

Miscellaneous notation

- A *hit* in a permutation is a sub-word which has the same order as a pattern in the replacement set.
- An *avoider* is permutation which contains no hits.
- A *trivial class* is an equivalence class containing only one permutation.
- The *parity* of a permutation is its signum/sign/oddness.

The three problems

1. **Rotations:** replacement sets containing a permutation and its rotations;
   Example: $\{2134, 1342, 3421, 4213\}$
2. **Single Set:** replacement sets containing permutations of length 3;
   Example: $\{123, 231, 321\}$
3. **Double Set:** two non-intersecting replacement sets containing permutations of length 3.
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How many classes are there?

Theorem

In S_n, there are always either 1 or 2 nontrivial classes created.

Examples for odd c:

- $\{123, 231, 312\}$ creates two non-trivial classes.
- $\{21345, 13452, 34521, 45213, 52134\}$ creates two non-trivial classes.

How many classes are there?

Theorem

In S_n, there are always either 1 or 2 nontrivial classes created.

Examples for even c:

- $\{1234, 2341, 3412, 4123\}$ creates one non-trivial class.
- $\{145236, 452361, 523614, 236145, 361452, 614523\}$ creates two non-trivial classes.

What if we rotate the identity?

Example: When $c = 5$, the replacement set is

$$\{12345, 23451, 34512, 45123, 51234\}$$

We consider only the non-trivial classes.

Only for odd c, there are two classes.

- For even n, they are the same size.
- For odd n, they differ in size.

Case of odd c, even n.

Definition

The rot of a permutation $x \in S_n$ is $(234\dots n1) \circ x$.

For example, $\mathrm{rot}(31524) = 42135$.

- rot preserves hits;
- Since n is even, rot changes parity;
- rot creates a bijection between odd non-avoiders and even non-avoiders.

Case of odd c, odd n: A main result

- Two classes: even non-avoiders and odd non-avoiders;
- Their sizes are multiples of n because of rot.

Theorem

The case $c > n/2$: The sizes of the two classes differ by $nC_{(n-c-2)/2}$, and the odd class is always larger.

$C_m = \frac{1}{m+1}\binom{2m}{m}$ is the mth Catalan number.

CASE OF ODD c, ODD n: OBSERVATION

Definition

A ***hit-ended*** *permutation has a hit only in the very beginning and very end.*

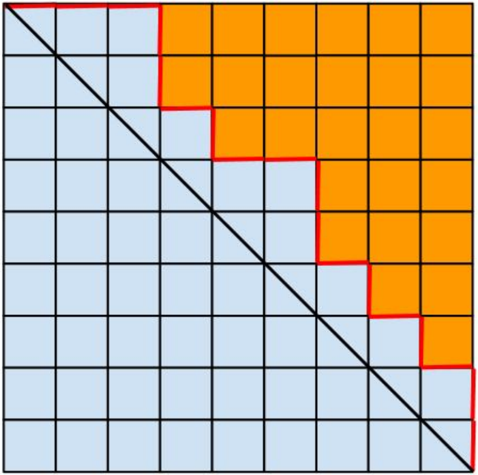
- Example: $n = 9$, $c = 3$, 815476392 is hit-ended.

$$\begin{aligned}&\text{(number of odd hit-ended permutations)}\\&\quad-\text{(number of even hit-ended permutations)}\\&=\\&\text{(number of even non-avoiding permutations)}\\&\quad-\text{(number of odd non-avoiding permutations)}\end{aligned}$$

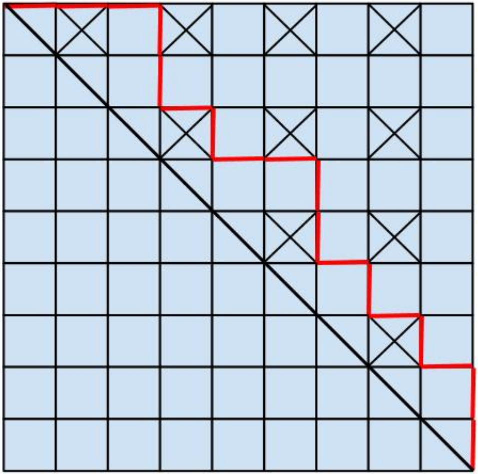
Case of odd c, odd n, $c > n/2$: reshaping the problem

- Because $c > n/2$, the two hits overlap.
- This allows a bijection between hit-ended permutations starting with a given letter and lattice paths inside an $(n - c - 1) \times (n - c - 1)$ square that stay above or on the diagonal.

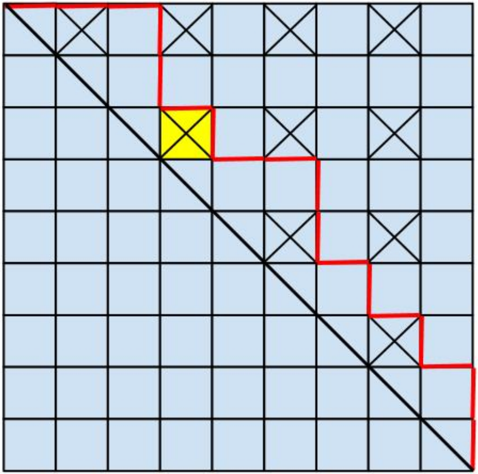
Solving the lattice problem: Looking at area



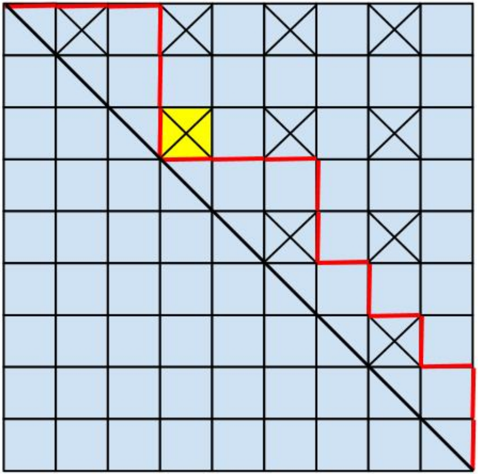
Solving the lattice problem: A partial bijection



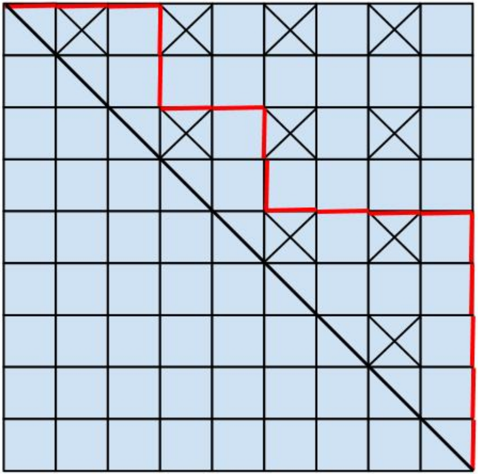
Solving the lattice problem: A partial bijection



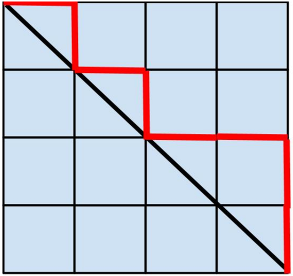
Solving the lattice problem: A partial bijection



Solving the lattice problem: The deciding paths



Solving the lattice problem: What's the enumeration?



There are $C_{(n-c-2)/2}$ lattice paths, so the sizes of the two classes differ by $nC_{(n-c-2)/2}$, and the odd class is always larger.

The three problems

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Past work on single replacement sets

- $\{123, 132, 213\}$, the Chinese relation
  - Number of classes in S_n = number of involutions in S_n (shown by Linton, Propp, Roby, and West).
- The number of classes was solved for some other cases by Pierrot, Rossin, and West.
- The number of permutations in the class containing the identity is known for all cases.

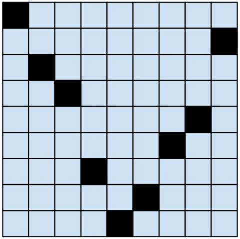
Single replacement set: Results

We prove formulas for the unsolved cases where the replacement set is of size > 2.

Replacement set	number of classes in S_n
$\{123, 132, 321\}$	$(n-1)!! + (n-2)!! + n - 2$
$\{123, 132, 312\}$	$f(n \geq 5) = f(n-1) + (n-2) \cdot f(n-2)$
$\{213, 231, 132\}$	$2^{n-2} + 2n - 4$
$\{123, 132, 231\}$	2^{n-1}
$\{123, 132, 213, 231\}$	n
$\{123, 132, 231, 321\}$	2 for $n > 3$
$\{213, 132, 231, 312\}$	3

$\{123, 132, 231\}$: PRELIMINARY

- A V-permutation is one that starts decreasing to 1 and then increases until the end.
  For example,



- In S_n there are 2^{n-1} V-permutations.

$\{123, 132, 231\}$: Reaching a V-permutation

- For $x \in S_n$ not starting with n, through repeated $132 \to 123$ and $231 \to 123$ we can place n as the final letter.
- So, n can always be moved to the start or end of a permutation.
- Inductively, every permutation is reachable from a V-permutation.
- There are at most 2^{n-1} classes.

$\{123, 132, 231\}$: What are the invariants?

Definition

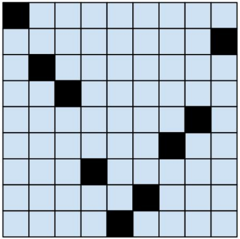
*A letter is **odd-tailed** if it's a left-to-right minimum and there are an even number of letters between it and the first left-to-right minimum to its right.*

Examples: 2 in 2134, 2 in 2341, 2 and 3 in 3214

Lemma

The set of odd-tailed letters in a permutation is invariant under the transformations.

$\{123, 132, 231\}$: What's the enumeration?



- In a V-permutation, each letter to the left of 1 is odd-tailed.
- No V-permutations are reachable from each other.
- There are 2^{n-1} classes.

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Past work on double replacement sets

As a restriction, each set is of size 2.

- Donald Knuth: $\{213, 231\}\{132, 312\}$ (*plactic relation*).
  - Number of classes in S_n = number of involutions in S_n $= f(n \geq 3) = f(n-1) + (n-1) \cdot f(n-2)$.
- Jean-Christophe Novelli and Anne Schilling: $\{213, 132\}\{231, 312\}$ (*forgotten relation*).
  - Number of classes in $S_n = n^2 - 3n + 4$.

Double replacement sets: Results

- We prove formulas for 9 of the 15 unsolved cases.

Replacement set	number of classes in S_n
$\{312, 321\}\{123, 132\}$	2^{n-1}
$\{123, 132\}\{213, 231\}$	2^{n-1}
$\{123, 231\}\{132, 321\}$	2^{n-1}
$\{132, 312\}\{321, 213\}$	$(n^2+n)/2-2$
$\{123, 231\}\{213, 132\}$	n^2-3n+4
$\{123, 132\}\{231, 312\}$	$3\cdot 2^{n-3}+n-2$ for $n>5$
$\{123, 132\}\{213, 321\}$	number of bushy-tailed permutations
$\{123, 321\}\{213, 231\}$	3 for $n>5$
$\{123, 132\}\{213, 312\}$	$f(n\geq 3)=f(n-1)+(n-1)\cdot f(n-2)$

$\{123, 132\}\{231, 312\}$: A slightly harder case

- Every permutation is reachable from a V-permutation;
- Are there 2^{n-1} classes?

$\{123, 132\}\{231, 312\}$: A slightly harder case

- Every permutation is reachable from a V-permutation;
- Are there 2^{n-1} classes?
- No! Two V-permutations x, y are reachable from each other iff the following is true:
  1. x and y have the same first letter.
  2. The letters directly preceding 1 in both x and y have value greater than 3.

$\{123, 132\}\{231, 312\}$: Adding up the classes

- 2^{n-2} classes with V-permutations $\ldots 21 \ldots$
- 2^{n-3} classes with V-permutations $\ldots 31 \ldots$
- $n-3$ classes with V-permutations $\ldots k1 \ldots$ where $k > 3$
- 1 class with V-permutation $1 \ldots$

There are $3 \cdot 2^{n-3} + n - 2$ classes in S_n.

Tips on studying replacement sets

- Write a C++ program;
  - Can calculate for n up to 12 in general case;
  - In case of rotations of identity, can calculate for cases up to $n = 23$;
  - Can run different calculations in just a couple of clicks.
- Examine the avoiding permutations separately.

Future Directions

- Solve the first problem for the remaining cases where $c < n/2$;
- Solve the second problem for sets of size 2;
- Solve the third problem for unsolved cases;
- Further investigate relations between third problem and Plactic relation.

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