Halving Lines and Underlying Graphs

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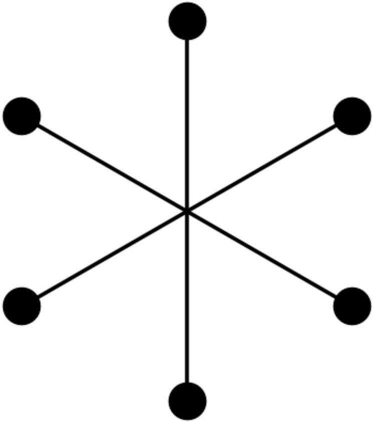
Background

Given n points, where n is even, a *halving line* is a line through two points that splits the other points in equal sets:

How many halving lines can a set of n points have?

Basic Facts

- Points on the convex hull have only one halving line
- Each point has an odd number of halving lines
- Affine transforms and dilations do not affect halving lines
- The minimum number of halving lines is $\frac{n}{2}$



Known Results

Current bounds for the maximum number of halving lines:

Theorem (Toth)

The maximum number of halving lines is at least $O(ne^{\Omega\sqrt{\log(n)}})$.

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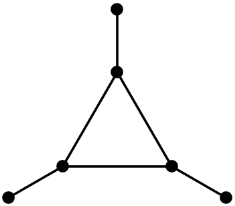
Theorem (Dey)

The maximum number of halving lines is at most $O(n^{\frac{4}{3}})$, or more precisely $\sqrt[3]{\binom{n}{2}\frac{4n^2}{135}}$.



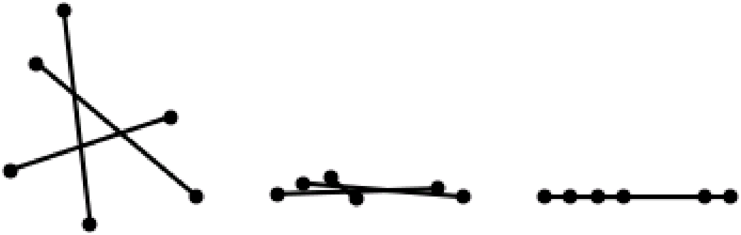
Constructions

Three special constructions:

- Segmenterizing
- Cross construction
- Y-shape construction

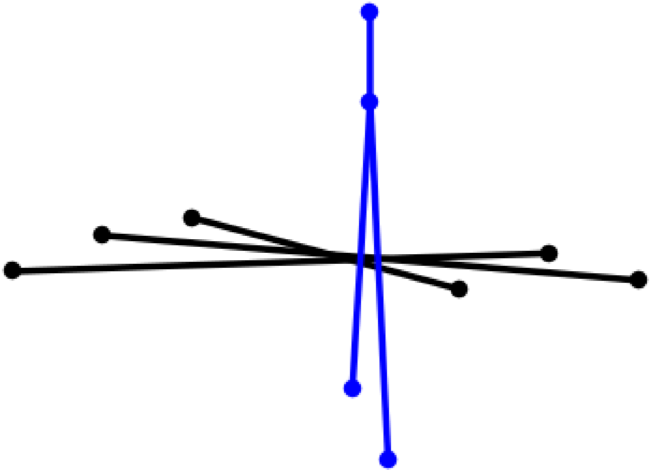
Segmentarize

- Squeeze along one direction until the points are nearly collinear



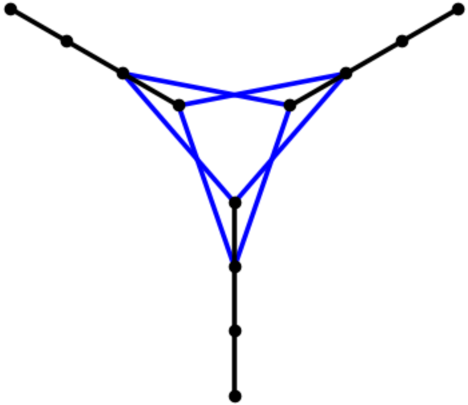
Cross

- Overlay two segmentarized configurations
- No additional halving lines



Y-shape

- Three segmenterized copies in a Y-shape
- Additional halving lines between branches



Underlying Graph

- Let the points be vertices and halving lines be edges

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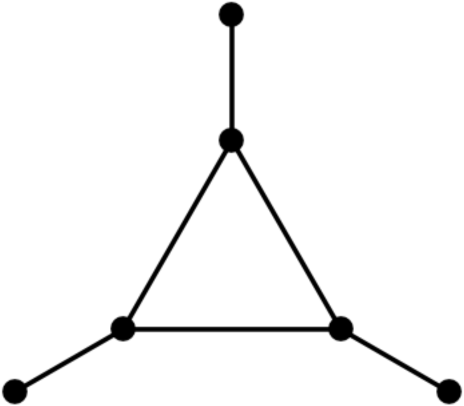
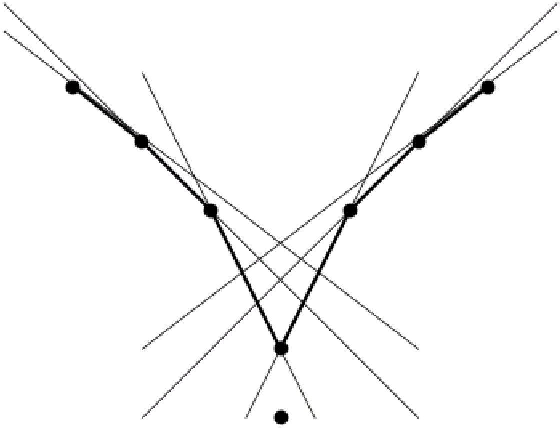
- Every vertex has odd degree



Path

Lemma

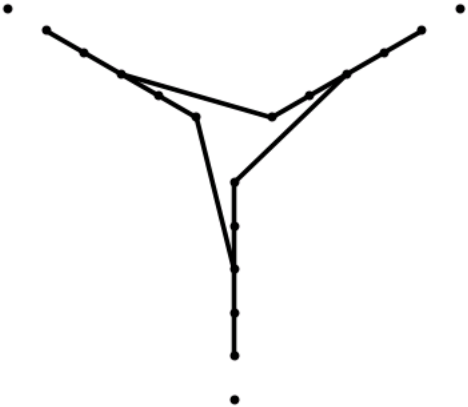
For every n, there exists an underlying graph with n vertices that contains a path of length $n-1$.



Cycles

Theorem

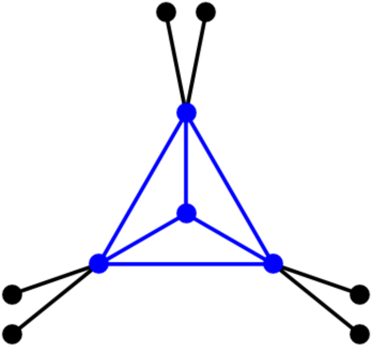
When n is a multiple of 6, the maximum length of a cycle is exactly $n - 3$.



Cliques

Theorem

The largest possible clique in an underlying graph of n vertices is at least $O(\sqrt{n})$.



New Bound

Lemma (Pach, Toth)

If a graph has E edges, V vertices, and crossing number C, and satisfies $E > \frac{15}{2}V$, then $C \geq \frac{135E^2}{4V^3}$.

Theorem

For sufficiently large n, the maximum number of halving lines is at most $\sqrt[3]{\binom{n}{2}\frac{n^2}{135}}$.

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- This improves Dey's bound of $\sqrt[3]{\binom{n}{2}\frac{4n^2}{135}}$ by a factor of $\sqrt[3]{4}$
- Since cliques have a quadratic number of crossings, an $O(n^{\frac{4}{3}})$ bound using the crossing lemma is optimal

Future Research

- Weighting vertices
- Better lower bound
- Other graph theoretic structures
- Algebraic structures

Acknowledgments

- Tanya Khovanova, for mentoring
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- PRIMES