Apollonian Equilateral Triangles

Christina Chen

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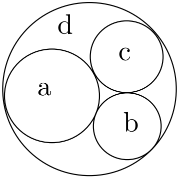
Geometric Motivation: Apollonian Circles



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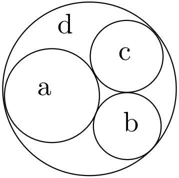

Figure 1: $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$.

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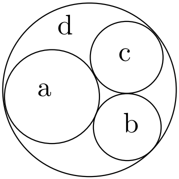

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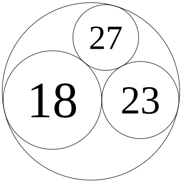

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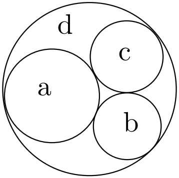

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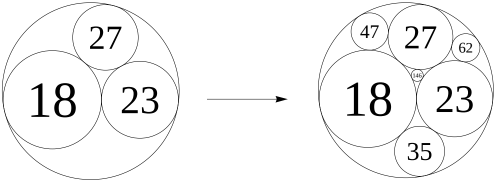

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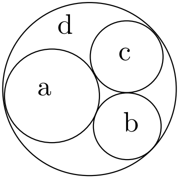

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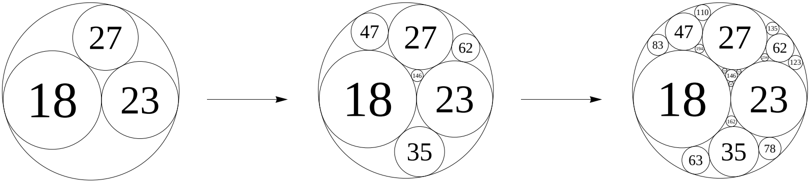

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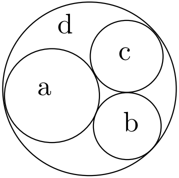

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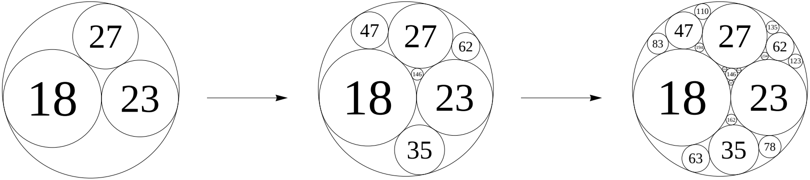
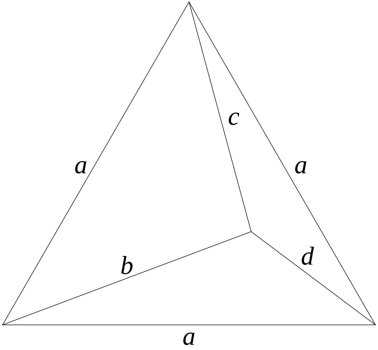


Figure 2: At each stage, a circle is incribed in each lune.

A Problem Involving an Equilateral Triangle



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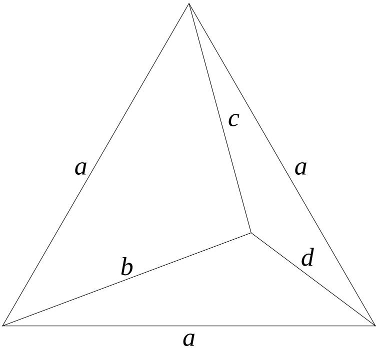

Figure 3: $(a + b + c + d)^2 = 3(a^2 + b^2 + c^2 + d^2)$.

Definitions

Definition (Triangle Quadruple)

A **triangle quadruple** $t = (a, b, c, d)$ is a quadruple of nonnegative integers satisfying

$$3(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

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Definition (Primitive Triangle Quadruple)

A triangle quadruple (a, b, c, d) is **primitive** if

$$gcd(a, b, c, d) = 1.$$

Operations

1) For solutions d and d' to the equation for triangle quadruples,

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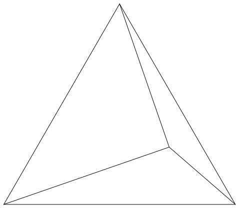
2) If (a, b, c, d) is a triangle quadruple, then

$$(a, b, c, a + b + c - d)$$

is also a triangle quadruple.

Geometric Representation of Operations



$t = (7, 4, 3, 1)$

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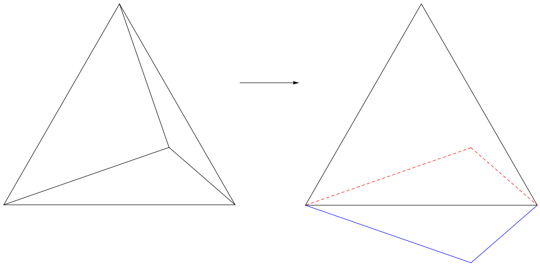

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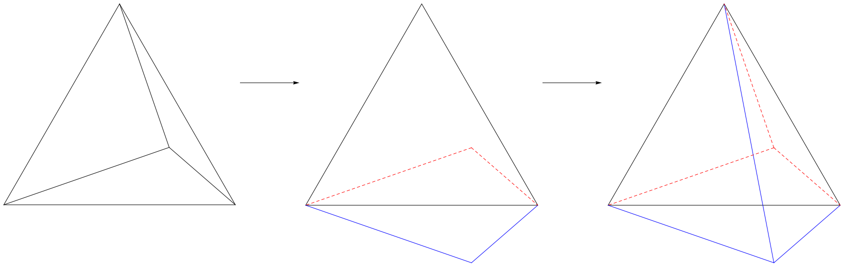

$t = (7, 4, 3, 1)$

$t' = (7, 4, 9, 1)$

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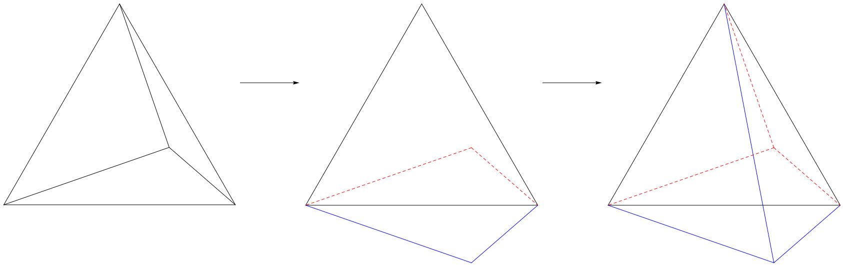

$t = (7, 4, 3, 1)$ $\qquad$ $t' = (7, 4, 9, 1)$

Figure 4: The operation is geometrically represented by reflecting two segments over a side of the equilateral triangle.

Matrix Representation of Operations

$$S_1 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

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For $\mathbf{v} = (a, b, c, d)^T$, $S_4\mathbf{v} = (a, b, c, a + b + c - d)^T$.

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Note that the generators satisfy:

1. $S_i^2 = I$ for $i = 1, 2, 3, 4$.
2. $(S_i S_j)^3 = I$ for $i \neq j$.

The Cayley Graph for the Triangle Group

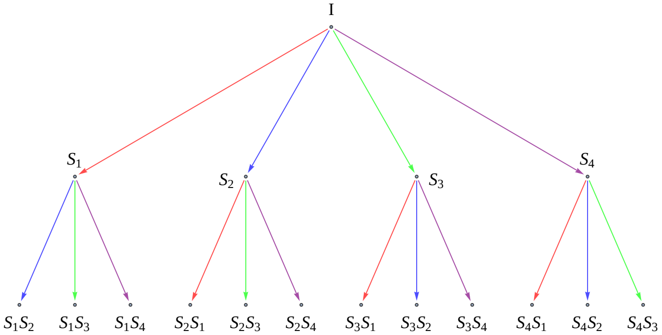

Figure 5: Part of the Cayley graph for the infinite triangle group.

Root Quadruples

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Lemma

For any triangle quadruple $t = (a, b, c, d)$*, operating on the largest element does not increase* $a + b + c + d$*.*

Root Quadruples

Lemma

Any triangle quadruple (a, b, c, d) *can be reduced to the root quadruple* $(0, x, x, x)$ *(or permutations), where* $x = gcd(a, b, c, d)$.

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Example

$(3, 4, 7, 1) \longrightarrow (3, 4, 1, 1) \longrightarrow (3, 1, 1, 1) \longrightarrow (0, 1, 1, 1)$

Consequences Involving Orbits

A triangle quadruple (a, b, c, d) can generate a triangle quadruple (a', b', c', d') in a finite number of operations if $gcd(a, b, c, d) = gcd(a', b', c', d')$.

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Theorem

All primitive triangle quadruples are contained in one orbit.

Counting the Number of Quadruples

Question

Is it possible to compute the number of triangle quadruples with height $\sqrt{a^2 + b^2 + c^2 + d^2}$ *below a given value?*

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Theorem

Let $F(x)$ *be the number of triangle quadruples with* $\sqrt{a^2+b^2+c^2+d^2} \leq x$*. Then* $F(x) = O(x^2)$*.*

Growth Rates

Growth Rates

Let W denote a word $S_{a_1} S_{a_2} \cdots$, where $S_{a_i} \neq S_{a_{i+1}}$.

Theorem

For any W of length $n \equiv i \pmod 4$ and a root quadruple $\mathbf{t} = (a, b, c, d)$ with $a \leq b \leq c \leq d$,

$$||W\mathbf{t}||_\infty \leq ||T_i(S_4 S_3 S_2 S_1)^{\frac{n-i}{4}}\mathbf{t}||_\infty,$$

where $T_i = I, S_1, S_2S_1, S_3S_2S_1$ for $i = 0, 1, 2, 3$, respectively.

Is the Triangle Group a Coxeter Group?

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The generators are reflections.

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Proof.

The eigenvalues of S_i are 1, 1, 1, -1. It follows that the operation corresponding to S_i is the reflection over the plane spanning the vectors v_{i_1}, v_{i_2}, v_{i_3}, denoting the eigenvectors of S_i. □

Is the Triangle Group a Coxeter Group?

Lemma

For $x = (a, b, c, d)$*,* S_i *preserves the quadratic form* $F(x) = 3(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 = xQx^T$*, where*

$$Q = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}.$$

That is, $F(x) = F(S_i x)$*.*

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Proof.

Abstractly, construct a Coxeter group with Q as its Cartan matrix. By the previous two lemmas, the triangle group is that Coxeter group. □

Open Questions

1. Beginning with a specific root quadruple, is it possible to calculate the average value of the maximum element in the triangle quadruple obtained after n operations?

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2. Given any integer n, is it possible to calculate the number of triangle quadruples with n as the largest element?

3. Given any pairs of number (p, q), is it possible to determine whether there exists a triangle quadruple containing p and q, and if such a quadruple does exist, is it possible to determine how many there are?

Acknowledgments

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- my parents, for always supporting me.

Bibliography

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