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The *Poisson algebra* $(A_0, \{,\})$ retains a great deal of information about the non-commutative family $A_\hbar$.
In particular, the *Poisson homology* HP_0 of A_0 gives an upper bound on the number of irreducible representations of the non-commutative family $A_\hbar$:

$$\#\text{Irreps}(A_\hbar) \leq \dim HP_0(A_0).$$

Poisson homology in characteristic p

Michael Zhang, Yongyi Chen
MIT PRIMES

May 21, 2011

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Example

$$\begin{aligned} \{xy, y^2\} &= x\{y, y^2\} + y\{x, y^2\} \\ &= 0 + y(2y\{x, y\}) \\ &= -2y^2. \end{aligned}$$

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Let S_2 act on $R = \mathbb{F}[x_1, x_2, y_1, y_2]$ by permuting indices (e.g. $(12) \cdot x_1 = x_2$). Then R^{S_2} is generated by the invariants $x_1 + x_2$, $y_1 + y_2$, x_1x_2, y_1y_2 and $x_1y_1 + x_2y_2$.

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Let $C_n = \langle g \mid g^n = 1 \rangle$ act on $R = \mathbb{F}[x, y]$ in the following way, where ω is a primitive nth root of unity:

$$g \cdot x = \omega x \quad \text{and} \quad g \cdot y = \omega^{-1} y.$$

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Then R^{C_n} is generated by x^n, y^n, and xy.

Problem statement and past results

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- We compute HP_0 when $\mathbb{F} = \mathbb{F}_p$. In this case, HP_0 is infinite-dimensional.

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- We form a grading

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- This is just a generating function with formal variable t formed from the grading.

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Theorem

If $G = Cyc_n$ *acts by* $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}$ *where* ω *is a primitive nth root of unity, for* $p > n$, $h(HP_0(A);t) = \sum_{m=0}^{n-2} t^{2m} + \frac{t^{2p-2}(1+t^{np})}{(1-t^{2p})(1-t^{np})}$

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For small p coprime with n, we prove a similar, but more complicated formula.

RESULTS FOR SUBGROUPS OF $SL_2(\mathbb{C})$

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Conjecture

For subgroups G of $SL_2(\mathbb{C})$, and $A = \mathbb{F}_p[x,y]^G$, the Hilbert series of $HP_0(A)$ is

$$h(HP_0(A);t) = \sum t^{2(m_i-1)} + t^{2(p-1)} \frac{1+t^h}{(1-t^a)(1-t^b)},$$

and a and b are degrees of the primary invariants.

Future Directions

- We will try to prove the afore-mentioned conjecture for subgroups of $SL_2(\mathbb{C})$. These are the dicylic group Dic_n and the exceptional groups E_6, E_7, E_8.

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- We will try to prove the afore-mentioned conjecture for subgroups of $SL_2(\mathbb{C})$. These are the dicylic group Dic_n and the exceptional groups E_6, E_7, E_8.
- The conjecture is a theorem already for large p. We will prove it for all $p > h$.
- We intend to extend our analysis of HP_0 to polynomial algebras of higher dimension, such as $\mathbb{F}[x_1, x_2, y_1, y_2]^G$.

Future Directions, cont.

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Conjecture

Let A be the algebra $\mathbb{F}_p[x,y,z]/Q(x,y,z)$ of functions on the cone X of a smooth plane curve of degree d (that is, Q is nonsingular, and homogeneous of degree d). Then,

$$h(HP_0(A);t) = \frac{(1-t^{d-1})^3}{(1-t)^3} + t^{p+d-3}f(t^p) \quad \textit{where}$$

$$f(z) = (1-z)^{-2}\left(2g - (2g-1)z + \sum_{j=0}^{d-2} z^j\right)$$

where $g = \frac{(d-1)(d-2)}{2}$ is the genus of the curve.

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- Thank you to our mentor, David Jordan, for being a great teacher, providing guidance and taking the significant time to help us out.