Fibonacci Numbers and Continued Fractions

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Continued Fractions

- $\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \dots}}}}$
- $\{3; 7, 15, 1, 292, \dots\}$
- Continued fractions are very useful in approximation theory.
- $\pi \approx \frac{22}{7}$

Continued Fractions

- $\{2; 1, 2, 1, 2, 1, \ldots\} = \{2; \overline{1, 2}\}$
- Rational $\iff$ Finite Continued Fraction
- Irrational $\iff$ Infinite Continued Fraction
- Quadratic Irrational $(a + b\sqrt{c}) \iff$ Periodic Infinite Continued Fraction
- Convergent: truncation of a continued fraction

Fibonacci Numbers

- $F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1$
- $\frac{F_{n+1}}{F_n} = \{\overbrace{1; 1, 1, 1, \ldots}^{n \text{ 1's}}\}$

  Proof: $\frac{F_{n+2}}{F_{n+1}} = \frac{F_{n+1}}{F_{n+1}} + \frac{F_n}{F_{n+1}} = 1 + \cfrac{1}{\cfrac{F_{n+1}}{F_n}}$
- Project Goal:

  Is there a pattern for the continued fraction of $\frac{F_{n+1}^m}{F_n^m}$?

Powers of the Golden Ratio

- $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1+\sqrt{5}}{2} \implies \lim_{n\to\infty} \frac{F_{n+1}^m}{F_n^m} = \phi^m$

$$\begin{array}{cc} \phi^1 = \{1; \overline{1}\} & \phi^2 = \{2; \overline{1}\} \\ \phi^3 = \{4; \overline{4}\} & \phi^4 = \{6; \overline{1, 5}\} \\ \phi^5 = \{11; \overline{11}\} & \phi^6 = \{17; \overline{1, 16}\} \end{array}$$

- L_n is the nth Lucas number. $L_{n+2} = L_{n+1} + L_n, L_0 = 2, L_1 = 1$

Theorem

- $\phi^n = \{L_n; \overline{L_n}\}$, *n is odd*
- $\phi^n = \{L_n - 1; \overline{1, L_n - 2}\}$, *n is even*

- The convergents of ϕ^n are $\frac{F_{mn+n}}{F_{mn}}$.

Squares

Examples:

- $\frac{F_5^2}{F_4^2} = \frac{25}{9} = \{2; 1, 3, 1, 1\}$
- $\frac{F_6^2}{F_5^2} = \frac{64}{25} = \{2; 1, 1, 3, 1, 1, 1\}$
- $\frac{F_7^2}{F_6^2} = \frac{169}{64} = \{2; 1, 1, 1, 3, 1, 1, 1, 1\}$

Theorem

$$\frac{F_{n+1}^2}{F_n^2} = \{2; \overbrace{1, 1, \ldots}^{n\text{-}3\ 1\text{'s}}, 3, \overbrace{1, 1, \ldots}^{n\text{-}2\ 1\text{'s}}\}$$

Cubes

Examples:

- $\frac{F_7^3}{F_6^3} = \{4; 3, 2, 3, 2, 2, 1, 1, 1\}$
- $\frac{F_8^3}{F_7^3} = \{4; 4, 1, 1, 1, 4, 2, 2, 1, 1, 1, 1\}$
- $\frac{F_9^3}{F_8^3} = \{4; 4, 10, 4, 2, 2, 1, 1, 1, 1, 1\}$
- $\frac{F_{10}^3}{F_9^3} = \{4; 4, 3, 2, 3, 4, 2, 2, 1, 1, 1, 1, 1, 1\}$
- $\frac{F_{11}^3}{F_{10}^3} = \{4; 4, 4, 1, 1, 1, 4, 4, 2, 2, 1, 1, 1, 1, 1, 1, 1\}$
- $\frac{F_{12}^3}{F_{11}^3} = \{4; 4, 4, 10, 4, 4, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1\}$

Cubes

Theorem

- $\frac{F_{3n+2}^3}{F_{3n+1}^3} = \{\overbrace{4; 4, 4, \ldots}^{n\ 4\text{'s}}, 1, 1, 1, \overbrace{4, 4, \ldots}^{n\text{-}1\ 4\text{'s}}, 2, 2, \overbrace{1, 1, \ldots, 1}^{3n\text{-}2\ 1\text{'s}}\}$
- $\frac{F_{3n+3}^3}{F_{3n+2}^3} = \{\overbrace{4; 4, 4, \ldots}^{n\ 4\text{'s}}, 10, \overbrace{4, 4, \ldots}^{n\text{-}1\ 4\text{'s}}, 2, 2, \overbrace{1, 1, \ldots, 1}^{3n\text{-}1\ 1\text{'s}}\}$
- $\frac{F_{3n+4}^3}{F_{3n+3}^3} = \{\overbrace{4; 4, 4, \ldots}^{n\ 4\text{'s}}, 3, 2, 3, \overbrace{4, 4, \ldots}^{n\text{-}1\ 4\text{'s}}, 2, 2, \overbrace{1, 1, \ldots, 1}^{3n\ 1\text{'s}}\}$

Fourth Power (Conjecture)

- $\frac{F_{63}^4}{F_{62}^4} =$

$$\{\underbrace{6;1,\overbrace{5,1,\ldots}^{\text{14 times}}}_{A},\underbrace{37}_{B},\underbrace{\overbrace{3,1,2,3,2,1,3,33,\ldots}^{\text{3 times (no 33 the last time)}}}_{C},\underbrace{10,1,10}_{D},\underbrace{\overbrace{11,\ldots}^{\text{23 times}}}_{E}\}$$

- For $\frac{F_{20n+a+1}^4}{F_{20n+a}^4}$:
- A consists of $5n$ repititions of 5,1 (the first is 6,1).
- B varies with a mod 4: $\{1\},\{37\},\{4,4,33\},\{6,1\}$.
- C consists of n repititions of 3,1,2,3,2,1,3,33 (sometimes the start or end is affected by B or D).
- D varies with a mod 5: $\{31,1,9\},\{10,1,10\},\{15\},\{33,3,1,2,2,1,1\},\{33,4,6\}$.
- E consists of $2\lfloor\frac{20n+a+2}{5}\rfloor$ repititions of 11, if $a \equiv 2 \pmod 5$, or $2\lfloor\frac{20n+a+2}{5}\rfloor - 1$ repititions of 11, otherwise.

*Red varies in length, while blue varies with a.

Fifth Power (Conjecture)

- $\frac{F_{17}^5}{F_{16}^5} = \{\underbrace{11; 11, 11}_{A}, \underbrace{3}_{B}, \underbrace{704915}_{C}, \underbrace{1, 1, 5}_{B}, \underbrace{11, 11}_{A}, \underbrace{21}_{D}, \underbrace{11, 11}_{A}\}$
- $\frac{F_{22}^5}{F_{21}^5} = \{11; 11, 11, 11, 2, 1, 86698886, 2, 5, 11, 11, 11, 21, 11, 11, 11\}$
- For $\frac{F_{5n+a+1}^5}{F_{5n+a}^5}$:
- A's consist of repititions of 11, whose lengths vary with $5n$.
- B's vary with a, but may be compacted into one term, depending on whether n is even or odd.
- D varies with a.

Fifth Power (Conjecture)

- C varies with a, but the value changes.
- For $a = 2$, the value is exceptionally large.
- $\frac{F_{17}^5}{F_{16}^5} = \{11; 11, 11, 3, 704915, 1, 1, 5, 11, 11, 21, 11, 11\}$
- $\frac{F_{13}^5}{F_{12}^5} = \{11; 11, 10, 1, 46137317, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 9, 11, 11\}$
- In addition, there is a series of 1's rather than 11's.

Future Research

- Fourth and fifth powers
- General theorem
- Polynomial of Fibonacci numbers
- Other Fibonacci-like sequences

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