Counting matrices with restricted positions by rank over finite fields

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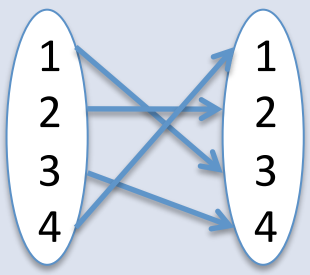
Motivation

objects

1. permutations

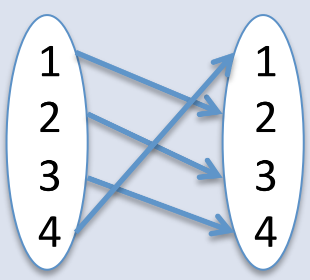
P_{3241}



0	0	**1**	0
0	**1**	0	0
0	0	0	**1**
1	0	0	0

2. permutations with **restricted** positions

P_{2341}



0	**1**	0	0
0	**0**	**1**	0
0	0	**0**	**1**
1	0	0	**0**

q-analogues

1'. invertible matrices over $\mathbf{F_q}$

0	1	1	0
1	0	0	1
0	0	2	1
1	0	1	1

2'. invertible matrices over $\mathbf{F_q}$ with **restricted** positions

0	2	1	0
0	**0**	2	0
1	0	**0**	2
1	1	2	**0**

Finite fields and rank of a matrix

$q = p^s$, F_q is the **finite field** with q elements
$s = 1$, F_q is the integers modulo q

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

x	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Rank of a matrix A is dimension of the row space of A
nxn matrix, det(A) ≠ 0 iff rank of A is n

Example

Rank 3

1	0	0
0	0	1
1	1	0

Rank 2

1	0	0
0	1	1
1	2	2

Rank 1

1	2	1
1	2	1
2	4	2

Rank 0

0	0	0
0	0	0
0	0	0

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Rank 3

1	0	0
0	0	1
1	1	0

1	0	0
1	0	1
0	3	1

Rank 2

1	0	0
0	1	1
1	2	2

Rank 1

1	2	1
1	2	1
2	4	2

Rank 0

0	0	0
0	0	0
0	0	0

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1	2	2

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1	2	1
2	4	2

Rank 0

0	0	0
0	0	0
0	0	0

1	0	0
1	0	1
0	3	1

F_3

1	0	0
1	0	1
0	0	1

Main counting problem: matrices over F_q with restricted positions

For ***S*** in *[n] x [n], let*

$$Mat_r(n, S) := \{A\ n \times n \text{ matrix over } \mathbf{F}_q \mid \text{rank } r, a_{ij} = 0 \text{ for } (i,j) \in S\}$$

Goal: Find *#Mat$_r$(n,**S**)*

(*i.e.* support of matrices **misses *S***)

Examples

1. For ***S*** = ∅, *r = n = 3*

(invertible matrices)

$$\#Mat_3(3, \emptyset) = \#GL(3,q)$$
$$= (q^3-1)(q^3-q)(q^3-q^2)$$
$$= (q-1)^3\mathbf{q^3}(\mathbf{q^2}+\mathbf{q}+\mathbf{1})(\mathbf{q}+\mathbf{1})$$

a_{11}	a_{12}	a_{13}
a_{21}	a_{21}	a_{23}
a_{31}	a_{32}	a_{33}

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$$= (q-1)^3 \mathbf{q^3(q^2 + q + 1)(q + 1)}$$

(invertible matrices)

0	a_{12}	a_{13}
a_{21}	**0**	a_{23}
a_{31}	a_{32}	**0**

3. For ***S*** = { (1,1), (2,2),…,(*n,n*) } , *r = n = 3*

(q-analogue of derangements)

$$\#Mat_3(3, S) = (q-1)^3 \mathbf{q(q^2 + 2q - 1)}$$

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$$\#Mat_3(3, S) = (q-1)^3 \mathbf{q}(\mathbf{q^2 + 2q - 1})$$

Questions

1. Is $\#Mat_r(n,S)$ **really** q-analogue of permutations with **restricted** positions ?

   Yes: $\#Mat_n(n, S)|_{q=1} \text{ “ = ” } \#\{\text{permutations avoiding } S\}$

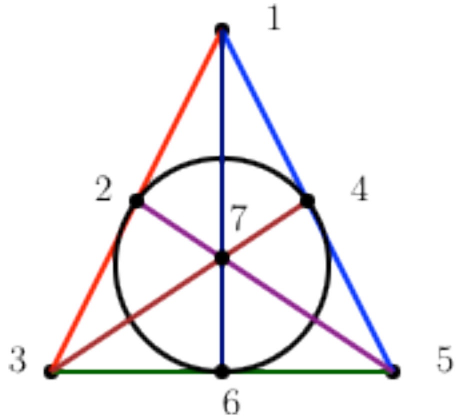
   (Liu-Lewis-Morales-Panova-Sam-Zhang 10)

2. How complicated is $\#Mat_r(n,S)$? Is it a **polynomial** in q?

#Mat$_r$(n,S) is not necessarily a polynomial in q

Example

Let $S_{PG(2,2)}$ in *[7]x[7]* be complement of support of the **Fano plane** *PG(2,2)*



PG(2,2)

A=

a_{11}	a_{12}	0	0	0	0	a_{17}
a_{21}	0	a_{23}	0	0	a_{26}	0
a_{31}	0	0	a_{34}	a_{34}	0	0
0	a_{41}	a_{42}	0	a_{44}	0	0
0	a_{52}	0	a_{54}	0	a_{56}	0
0	0	a_{63}	a_{64}	0	0	a_{66}
0	0	0	0	a_{75}	a_{76}	a_{77}

 # $S_{PG(2,2)}$ = 28

#Mat$_r$(n,S) is not necessarily a polynomial in q

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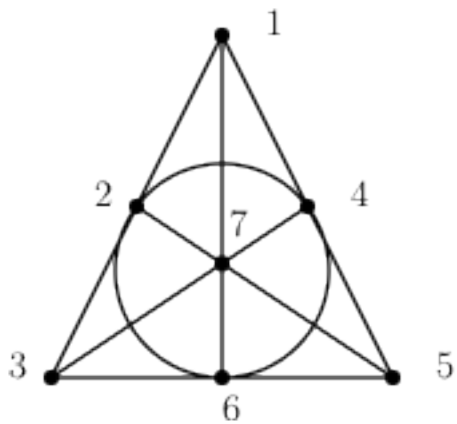

PG(2,2)

***A*=**

a_{11}	a_{12}	0	0	0	0	a_{17}
a_{21}	0	a_{23}	0	0	a_{26}	0
a_{31}	0	0	a_{34}	a_{34}	0	0
0	a_{41}	a_{42}	0	a_{44}	0	0
0	a_{52}	0	a_{54}	0	a_{56}	0
0	0	a_{63}	a_{64}	0	0	a_{66}
0	0	0	0	a_{75}	a_{76}	a_{77}

$\# S_{PG(2,2)} = 28$

Theorem (Stembridge 98)

The # invertible matrices ***A*** is the **quasi-polynomial:**

$$\#Mat_7(7, S_{PG(2,2)}) = \begin{cases} (q-1)^7 q^3 (q^{11} + \cdots - 97q^6 + \cdots + 1) & \text{if } q \text{ even}, \\ (q-1)^7 q^5 (q^9 + \cdots - 98q^4 + \cdots - 6) & \text{if } q \text{ odd}. \end{cases}$$

$S_{PG(2,2)}$ **smallest** example with respect to *n* and *#S*

In general $\#Mat_r(n,S)$ can be very hard

Polynomiality of $\#Mat_r(n,S)$ is related to a speculation of Kontsevich:

Conjecture (Kontsevich 97, Stanley's reformulation 98)

Let G is simple connected graph on n vertices, $S_G = \{ (i,j) \mid i \neq j, (i,j) \text{ in } E(G)\}$
then $\#\{A \text{ in } Mat_n(n,S_G) \text{ and symmetric}\}$ is a polynomial in q.

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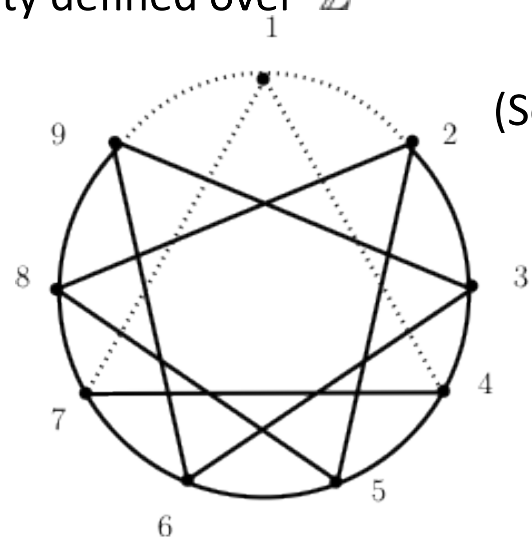
- Conjecture is **false,** $\#\{A \text{ in } Mat_n(n,S_G)$ *and* symmetric$\}$ as complicated as counting points over $\boldsymbol{F_q}$ of any variety defined over $\mathbb{Z}$ (Belkale-Brosnan 00)

- First counterexamples: G with E(G)=14. (Schnetz 09)

$\#\{A \text{ in } Mat_n(n,S_G)$ *and* symmetric$\}$ *three polynomials depending on* $q \equiv 0, 1, 2 \pmod 3$

G=



Goal: Find $\#Mat_r(n,S)$

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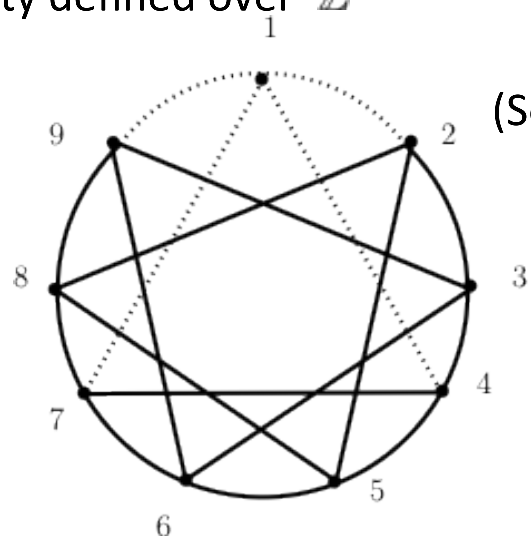


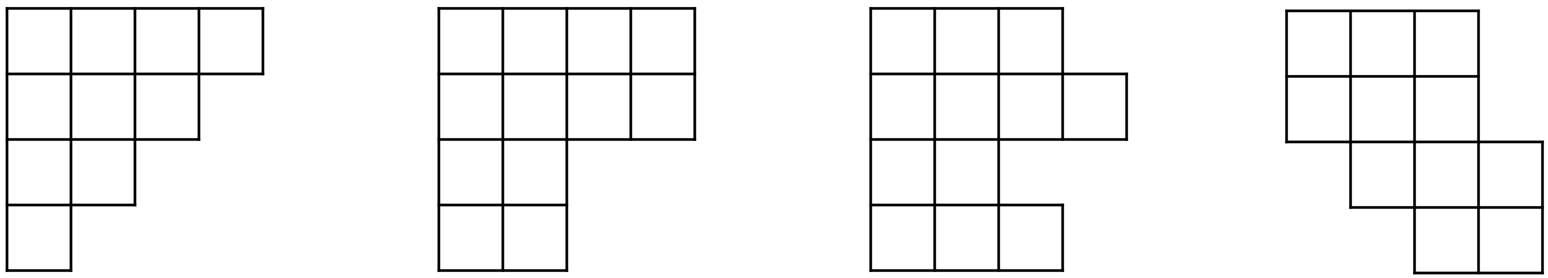
~~**Goal:** Find $\#Mat_r(n,S)$~~

New goal: Find families of sets S where $\#Mat_r(n,S)$ *is nice*

Straight and skew shapes

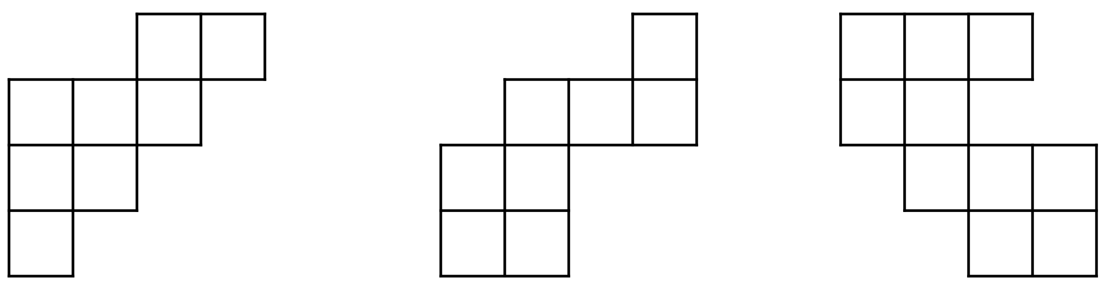
Straight shape or Young diagram – number of cells non-increasing, left-justified
Number of cells per row λ ($\lambda_1 \geq \lambda_2$, etc.), shape S_λ

A [4 3 2 1] B [4 4 2 2] C [3 4 2 3] D



Skew shape or skew Young diagram
Take one straight shape S_λ,remove another straight shape S_μ from upper left

A [4 3 2 1]/[2] B [4 4 2 2]/[31] C

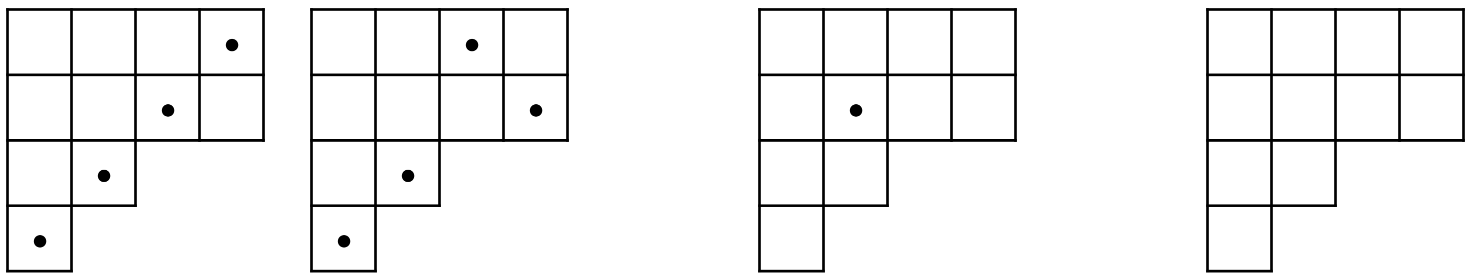


Rook placements and inversions

r-**rook placement**: *r* dots, "rooks"

No row or column has more than one rook

r = 4 r = 1 r = 0

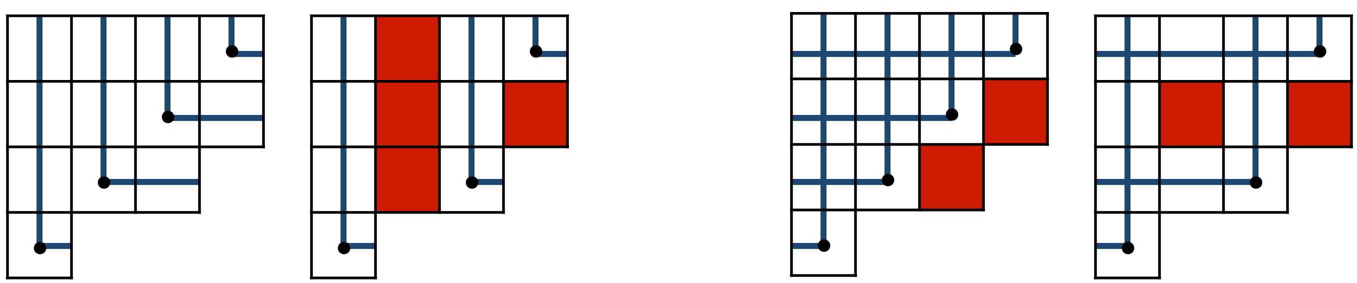


Inversions of a rook placement *c*: ***inv*(*c*)**

Number of cells in *S* that are not in line with any of the rooks in certain directions

Either NE-inversion or NW-inversion

NE NW



NE-inv(c) = 0 NE-inv(c) = 4 NW-inv(c) = 2 NW-inv(c) = 2

$\#Mat_r(S,q)$ for straight shapes S_λ

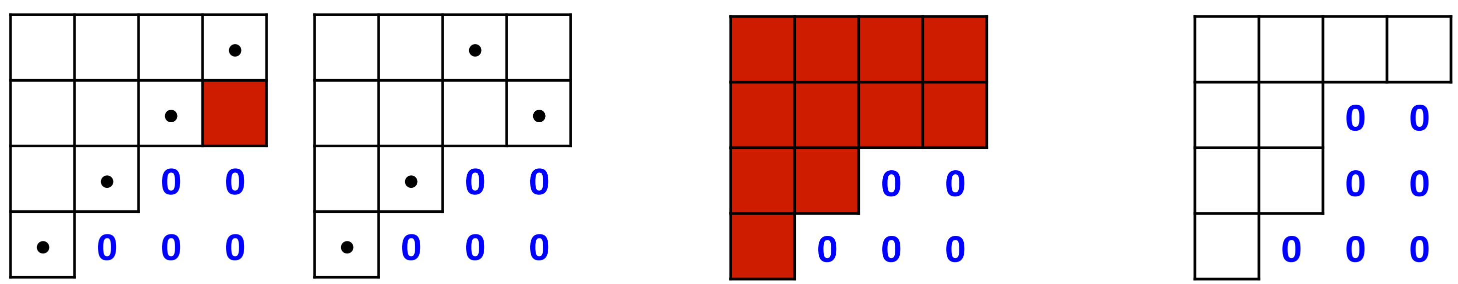
Theorem (Haglund 98)

For straight shapes $S = S_\lambda$

$$\#Mat_r(n, \overline{S}) = (q-1)^r q^{\#S - r} \sum_{\text{rook placements } C} q^{-(\text{NW-inv}(C,S))}$$

Example

r = 4 r = 0 r = 4



$(q\text{-}1)^4 q^7 (1 + {}^1\!/_q)$ $(q\text{-}1)^0 q^{11} q^{-11} = 1$ $(q\text{-}1)^4 q^5 0 = 0$

Note: for straight shapes S_λ the sum of $q^{-NW\text{-}inv(C,S)}$ is the same for NE-inversions and NW-inversions.

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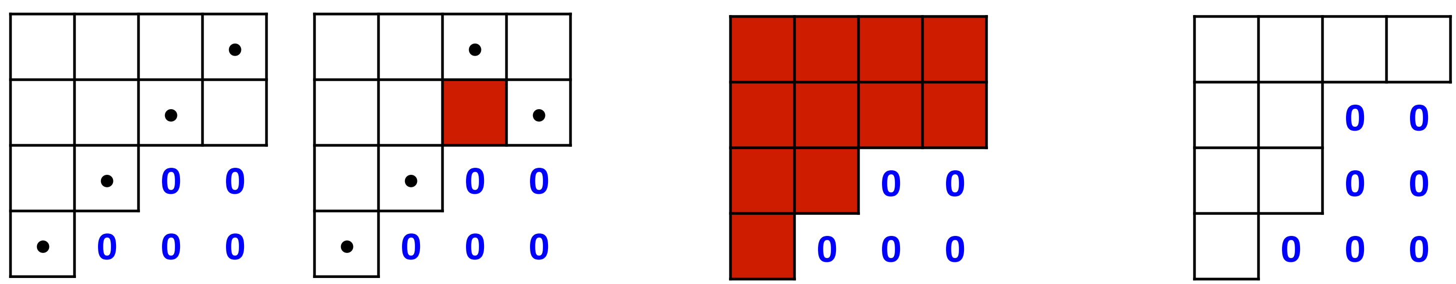
Example

r = 4 | r = 0 | r = 4



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Note: for straight shapes S_λ the sum of $q^{-NW\text{-}inv(C,S)}$ is the same for NE-inversions and NW-inversions.

Main Result: $\#Mat_r(S,q)$ for skew shapes $S_{\lambda/\mu}$

Haglund's type formula holds for skew shapes with **NE**-inv rather than **NW-**inv

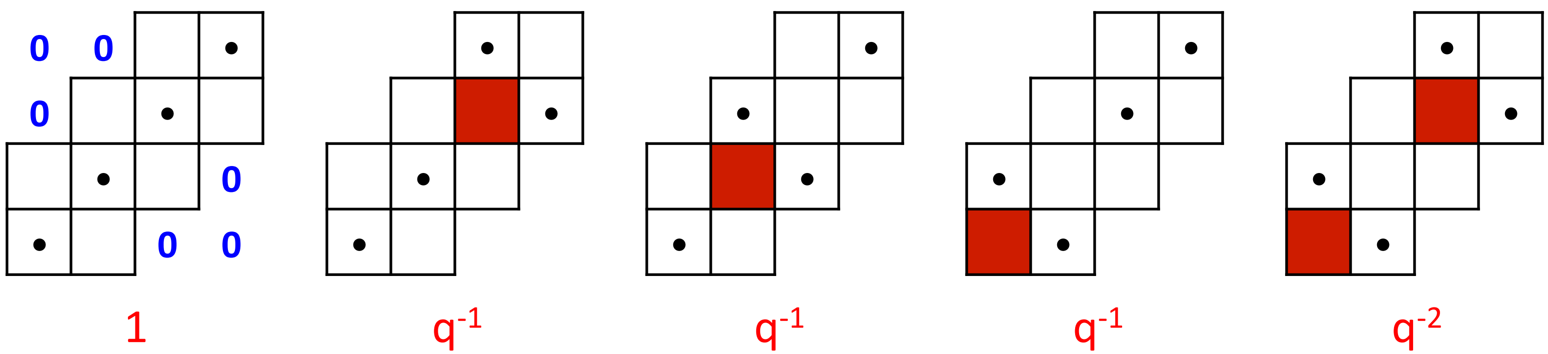
Theorem

For skew shapes $S=S_{\lambda/\mu}$

$$\#Mat_r(n,\overline{S}) = (q-1)^r q^{\#S-r} \sum_{\text{rook placements } C} q^{-(\text{NE-inv}(C,S))}$$

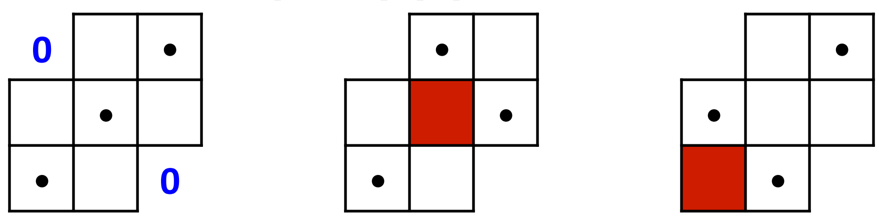
Example

[4 4 3 2]/[2 1] rank 4



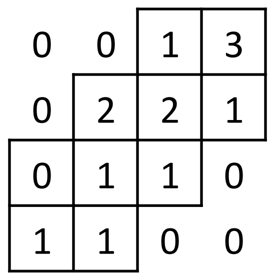
$\#T = (q-1)^4 q^6 (1+{}^3/_q+{}^1/_{q^2}) = \mathbf{(q-1)^4 q^4 (q^2+3q+1)}$

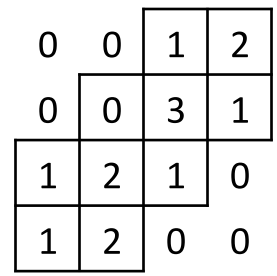
[3 3 2]/[1] rank 3



$\#T = (q-1)^3 q^4 (1+{}^2/_q) = \mathbf{(q-1)^4 q^3 (q+2)}$

Proof





Proof

0	0	1	3
0	2	2	1
0	1	1	0
1	1	0	0

0	0	1	2
0	0	3	1
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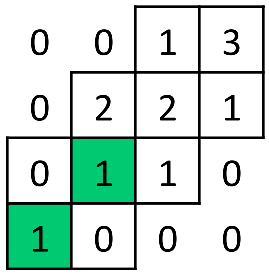

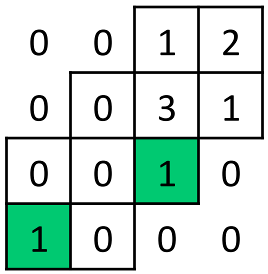

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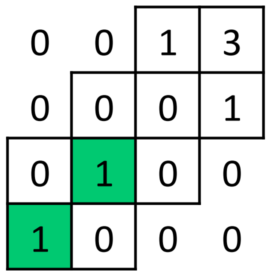
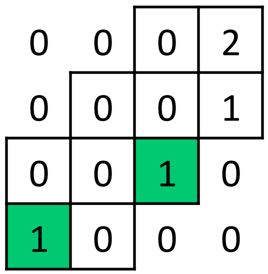


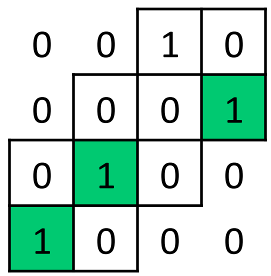


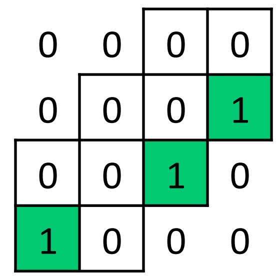
Proof

0	0	1	3
0	0	0	1
0	1	0	0
1	0	0	0

0	0	0	2
0	0	0	1
0	0	1	0
1	0	0	0

Proof





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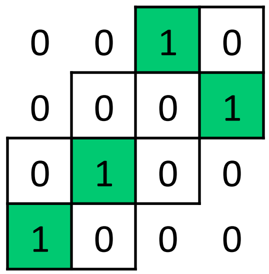

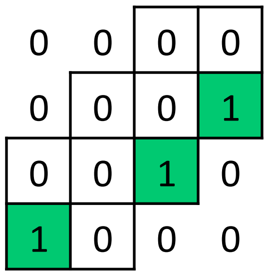

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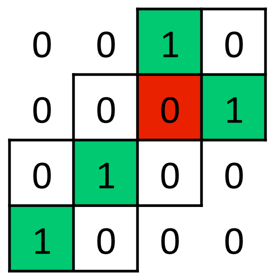
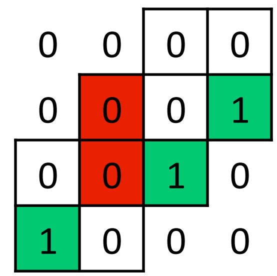
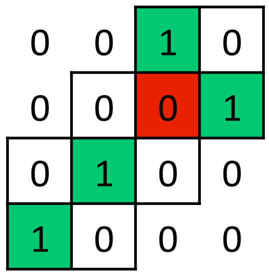
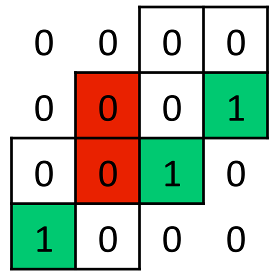




How many matrices correspond to each rook placement?

Proof





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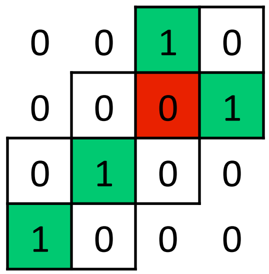
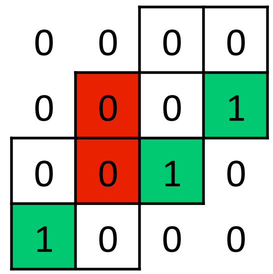
$(q-1)^r$

Proof

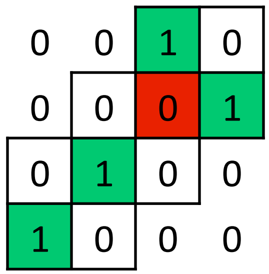


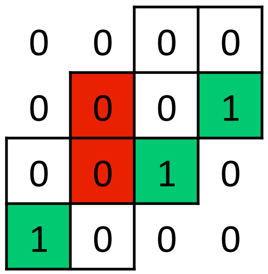


How many matrices correspond to each rook placement?

$(q-1)^r q^{\text{boxes eliminated}}$

Proof





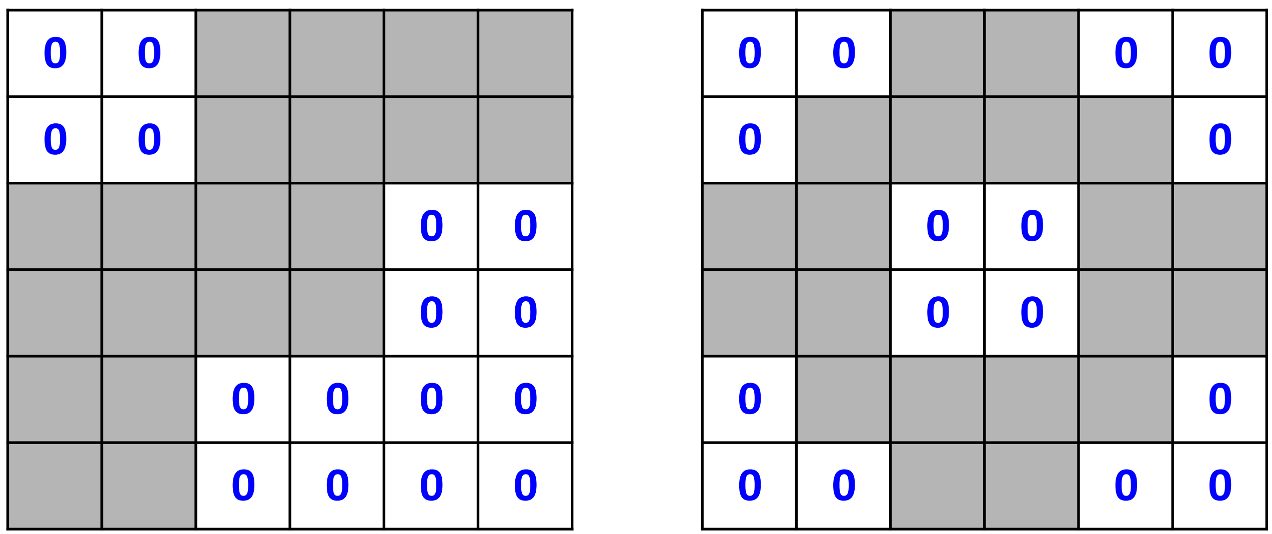
How many matrices correspond to each rook placement?

$$(q-1)^{r}q^{\#S-r-\mathrm{inv}(c)}$$

Permuting rows and columns in S

Recall: $\#Mat_r(n,S)$ is **invariant** under permuting rows and columns.

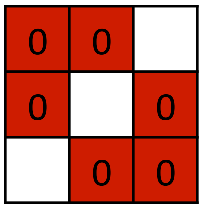
Example



Future work

1. Inverse skew-shapes: Haglund type formulas do not give $\#Mat_r(n,S)$

Example



Future work

1. Inverse skew-shapes: Haglund type formulas do not give *#Mat*$_r$*(n,S)*

Example

		0
	0	
0		

Future work

1. Inverse skew-shapes: Haglund type formulas do not give $\#Mat_r(n,S)$

Example

		0
	0	
0		

rank 3: $(q-1)^3(q^3+2q^2-q)$
rank 2: $(q-1)^2(q^3+6q^2+3q-1)$
rank 1: $(q-1)(6q)$
rank 0: 1

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0		

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Conjecture (LLMPSZ 10) $\#Mat_r(n, S_{\lambda/\mu})$ is a polynomial in q.

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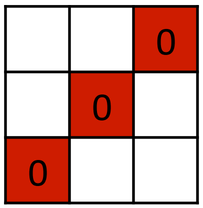

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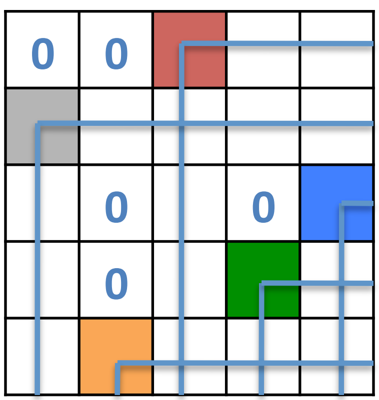
2. Rothe diagrams of permutations: S_w for permutation w

Example w=31542



rank 5: $(q-1)^5 (q+1)(q^2+2q^2+3q+1)$

Conjecture $\#Mat_r(n, \overline{S_w}) \times (q-1)^{-r}$ is polynomial with **non-negative** coefficients.

$$\begin{bmatrix} 0 & 0 & 0 \\ & 0 & \\ & 0 & \\ & 0 & \end{bmatrix} \begin{bmatrix} 0 & & 0 \\ 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & 0 \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} 0 & & & 0 \\ 0 & 0 & & 0 \\ 0 & & 0 & 0 \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} 0 & & \\ 0 & & 0 \\ 0 & 0 & \\ 0 & & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & & 0 \\ 0 & 0 & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & & 0 \\ 0 & & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & & 0 \\ 0 & & 0 \\ 0 & & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Professor Alexander Postnikov for proposing the problem
- My mentor Alejandro Morales
- PRIMES organizers and sponsors for setting up the PRIMES program and enabling me to give this talk