Progress on Parallel Chip-Firing

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MIT PRIMES

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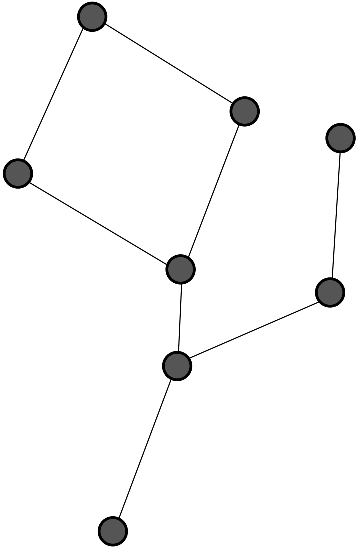
Motivation

- Simple rules
- "Obvious" patterns which are difficult to prove, or even wrong
- Potential connections to other fields of mathematics and science

Graphs



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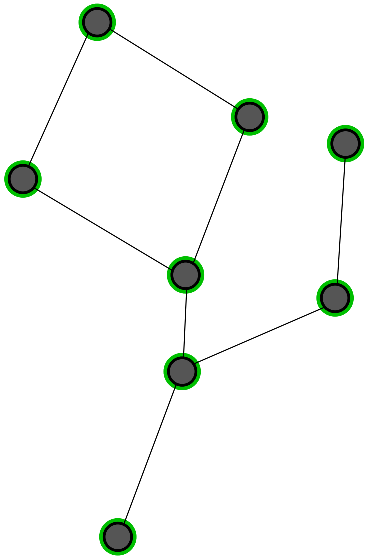

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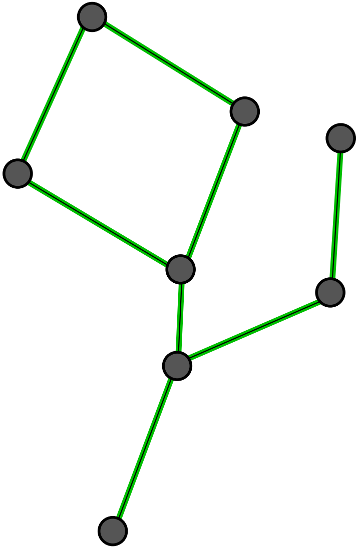

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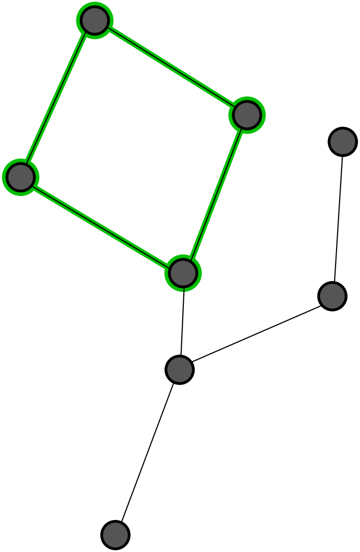

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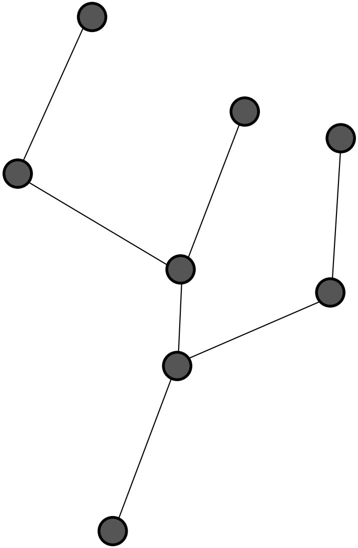
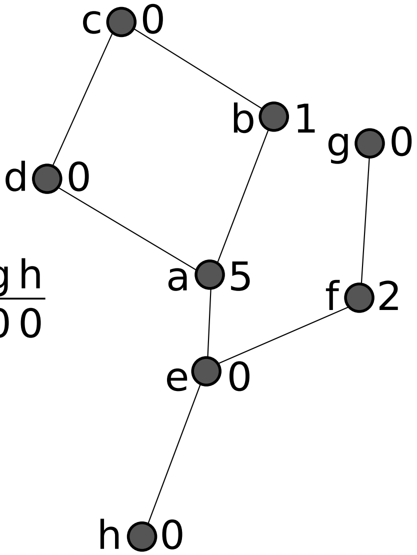


The Parallel Chip-Firing Game

- Played on a graph
- Assign a number of chips to each vertex
- On each turn:
  - If a vertex has at least as many chips as neighbors, it *fires*
    - Otherwise, we say it *waits*
  - When a vertex fires, it gives one chip to each of its neighbors
  - Happens for all vertices in parallel

The Parallel Chip-Firing Game

a	b	c	d	e	f	g	h
5	1	0	0	0	2	0	0



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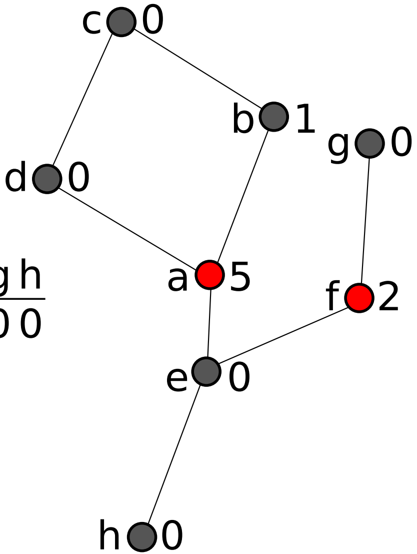

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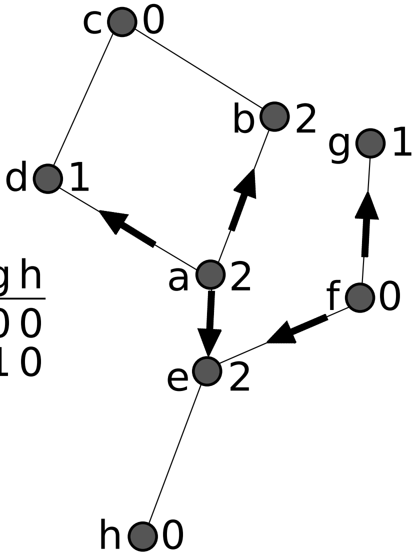
2	2	0	1	2	0	1	0



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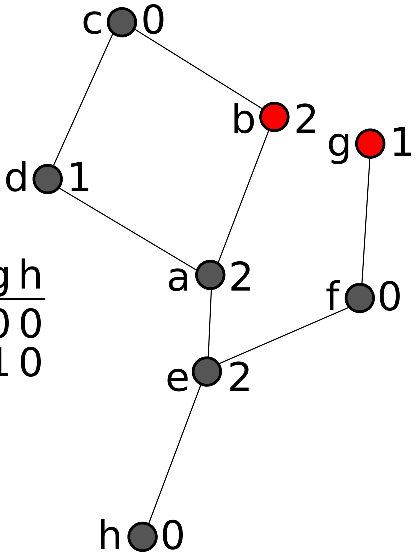

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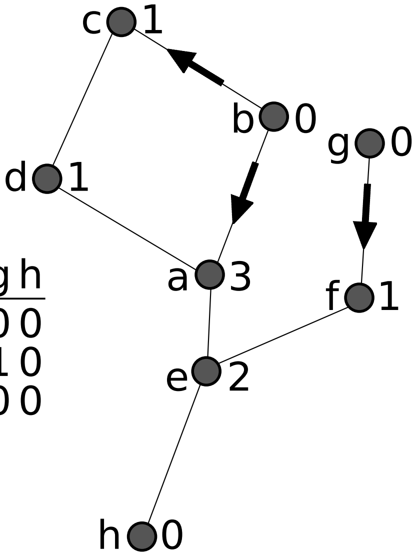
3	0	1	1	2	1	0	0



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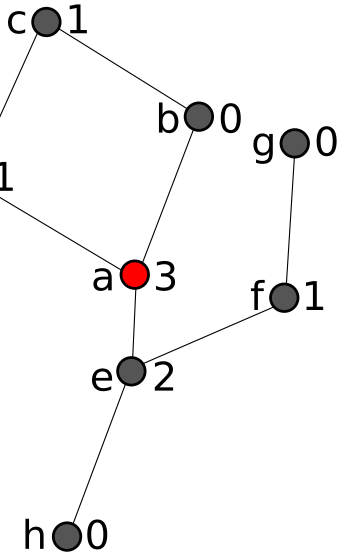

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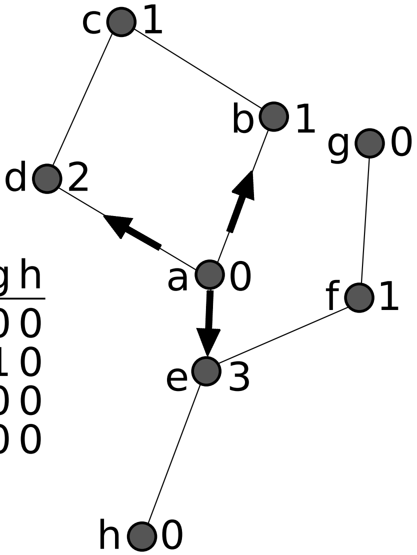

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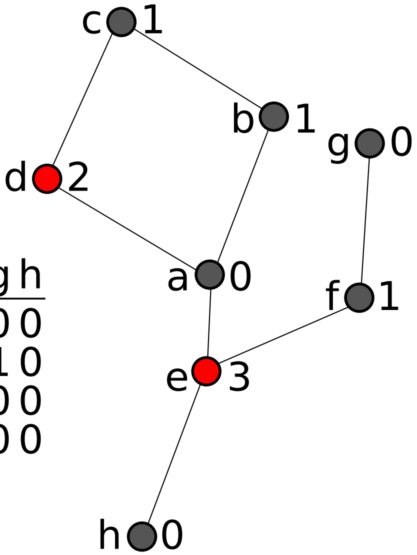

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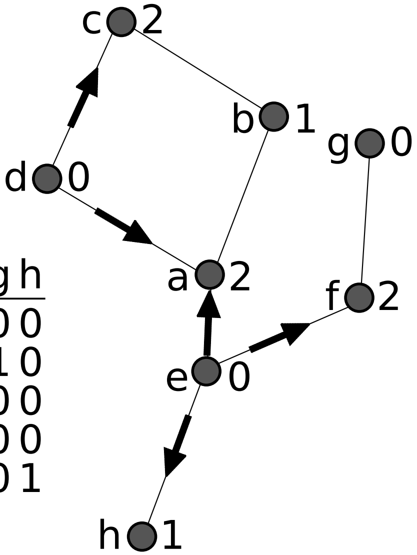

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0	1	1	2	3	1	0	0
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The Parallel Chip-Firing Game

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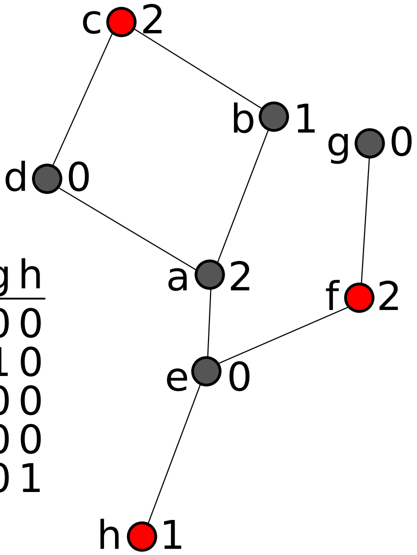

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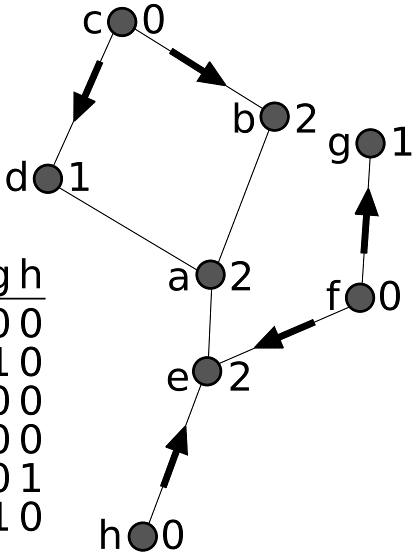

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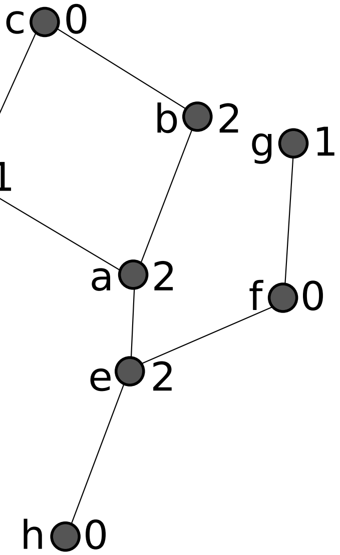

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0	1	1	2	3	1	0	0
2	1	2	0	0	2	0	1
2	2	0	1	2	0	1	0

Basic Properties

- All games are eventually periodic
- All vertices fire the same number of times in a period
  - In a periodic-1 position, either all vertices fire or all vertices wait
- Period > 2 needs a cycle

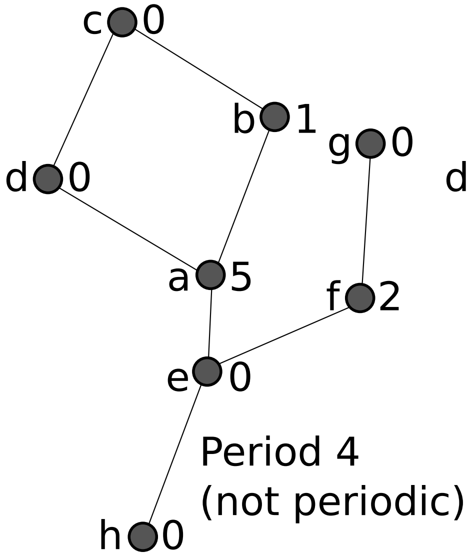
Notation

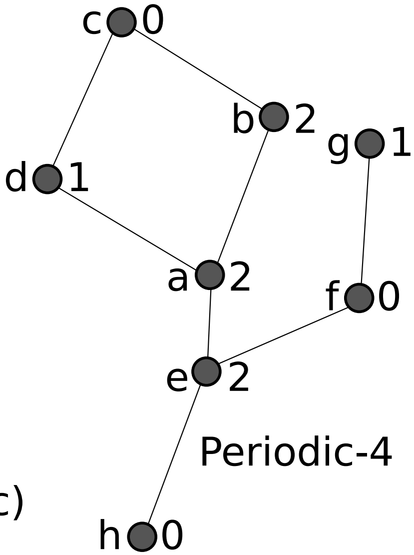
- $\sigma(t)$ is the position after taking t turns, starting with position $\sigma(0)$
- $\sigma_v(t)$ is the number of chips on vertex v in position $\sigma(t)$
- $\Phi_v(t)$ is the number of v's neighbors that fire at time t; v gets one chip from each
- $F_v(t)$ is 1 if v fires at time t and 0 otherwise
- c is the total number of chips in a position
- If G is a graph, $V(G)$ is its vertex set and $E(G)$ is its edge set

Outline of Literature

- Bitar's conjecture: maximum period $\leq$ number of vertices
- Bitar and Goles: Trees have period 1 or 2
- Kiwi et al.: Bitar's conjecture is false!
- Dall'Asta: Period on C_n divides n
- Levine: Period on $K_n \leq n$
- Jiang: Period on $K_{a,b} \leq 2\min(a, b)$

Periodic or Not?





Periodic-2 Positions

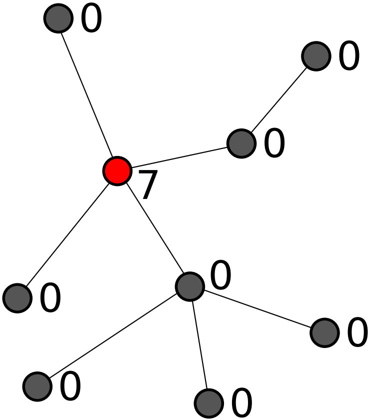
Theorem (Characterization of periodic-2 positions)

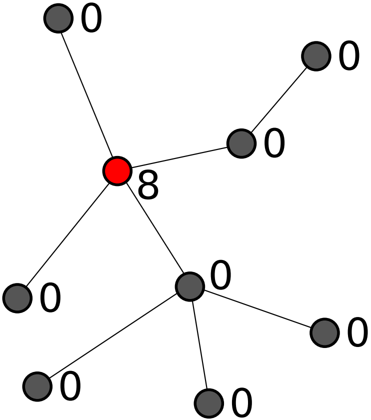
A position $\sigma(t)$ *on graph* G *is periodic-2 if and only if for all* $v \in V(G)$,
$\deg(v) \leq \sigma_v(t) + \Phi_v(t) \leq 2\deg(v) - 1.$

Proof.

When the period is 2, vertices alternate between firing and waiting. The above inequality is true if and only if v is about to switch states. □

Understanding Trees





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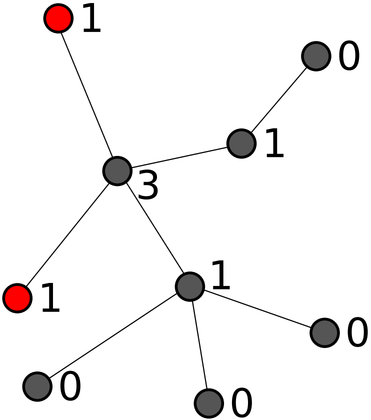

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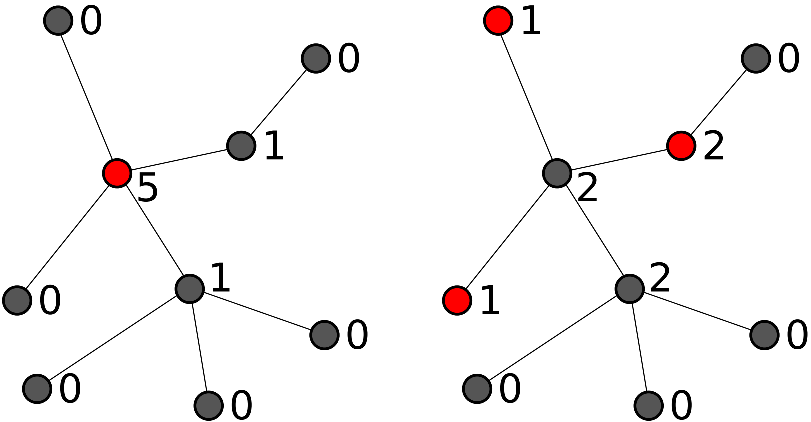

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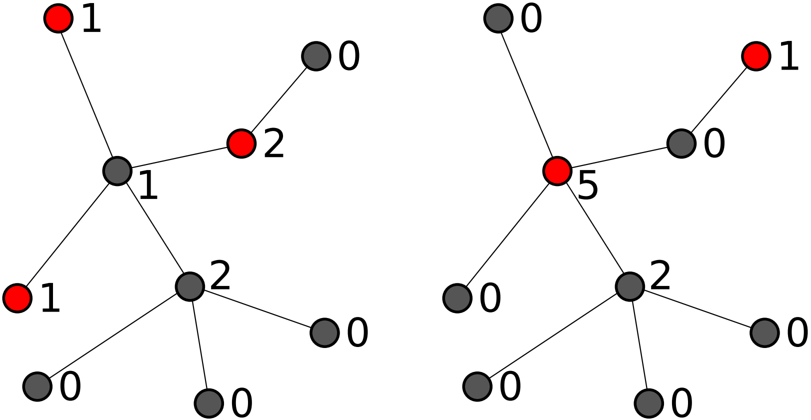

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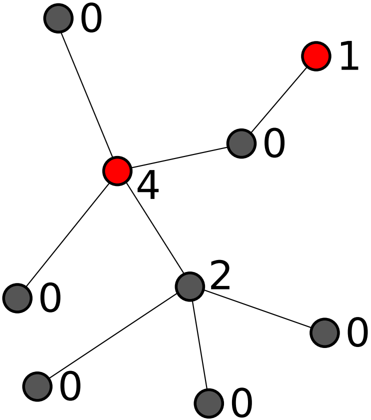

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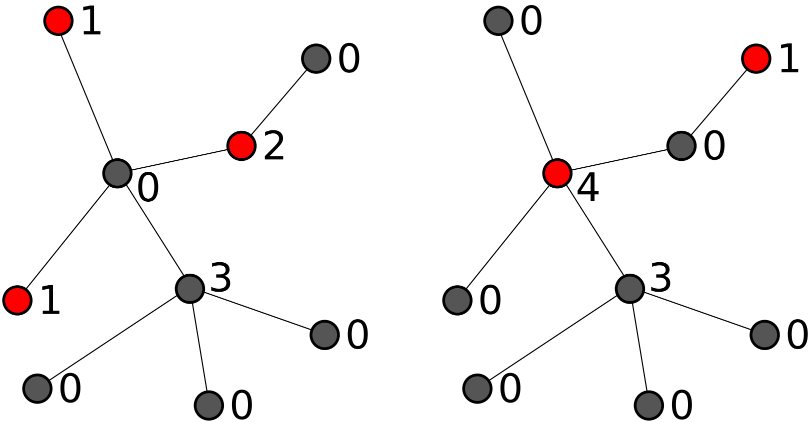

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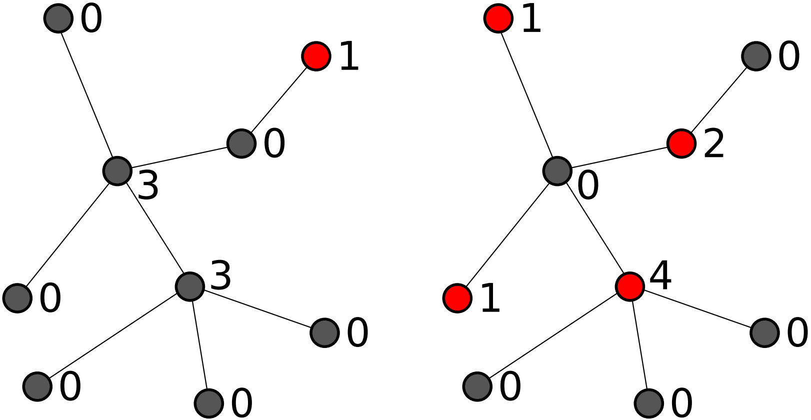

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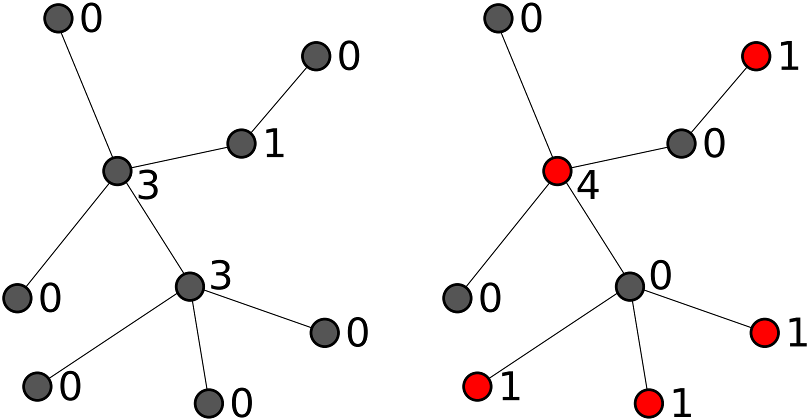

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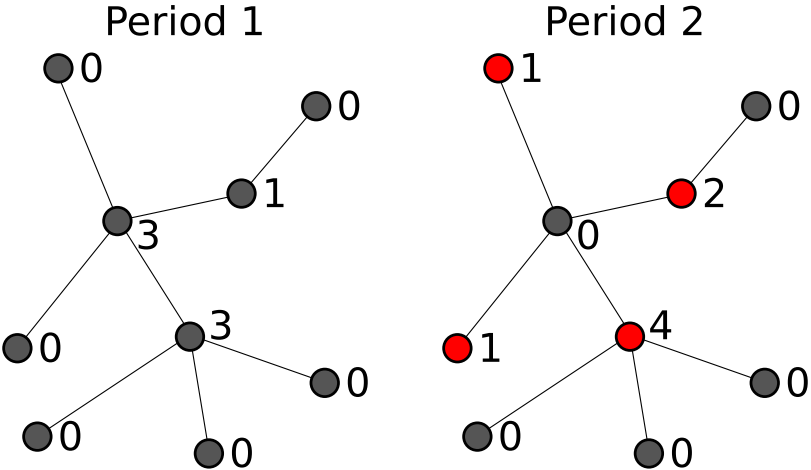

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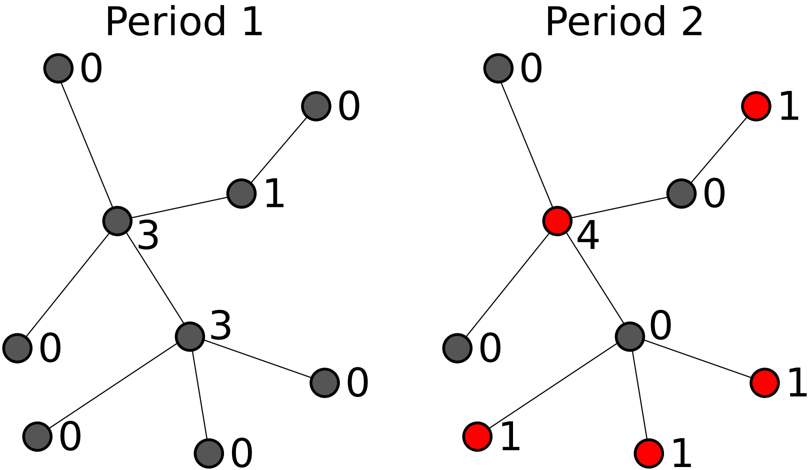

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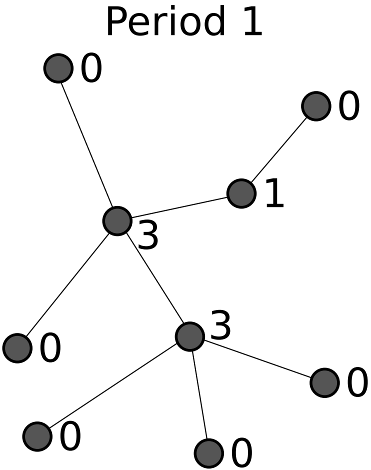

Understanding Trees

Theorem (Number of chips on a tree determines period)

If a game on a tree graph G has c chips, its eventual period is 2 if and only if $|E(G)| \leq c \leq 2|E(G)| - 1$.

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If a game on a tree graph G has c chips, its eventual period is 2 if and only if $|E(G)| \leq c \leq 2|E(G)| - 1$.

Proof.

If the period is n, then for some time t, $\sigma(t)$ will be periodic-n.

If $n = 1$:

$$\sigma_v(t) \leq \deg(v) - 1 \qquad\qquad \deg(v) \leq \sigma_v(t)$$

$$c \leq |E(G)| - 1 \qquad\qquad 2|E(G)| \leq c$$

If $n = 2$:

$$\deg(v) \leq \sigma_v(t) + \Phi_v(t) \leq 2\deg(v) - 1$$

$$2|E(G)| \leq c + \sum \frac{\Phi_v(t) + \Phi_v(t+1)}{2} \leq 3|E(G)| - 1$$

$$|E(G)| \leq c \leq 2|E(G)| - 1$$

□

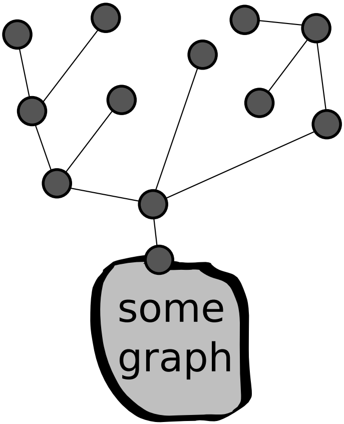
Firing Patterns

- String of 1s and 0s indicating firing and waiting, respectively
- Classification
  - Alternating: $(1, 0)$
  - Sparse: not alternating, two types
    - Sparsely firing: never fires twice in a row
    - Sparsely waiting: never waits twice in a row
  - Clumpy: neither sparse nor alternating

Motors

- A special vertex with a fixed firing pattern
- Doesn't care about receiving chips
- Natural motors
  - Subgraphs that follow normal chip firing rules
  - One key vertex behaves like a motor
    - Receiving external chips doesn't change its firing pattern

Motorized Trees



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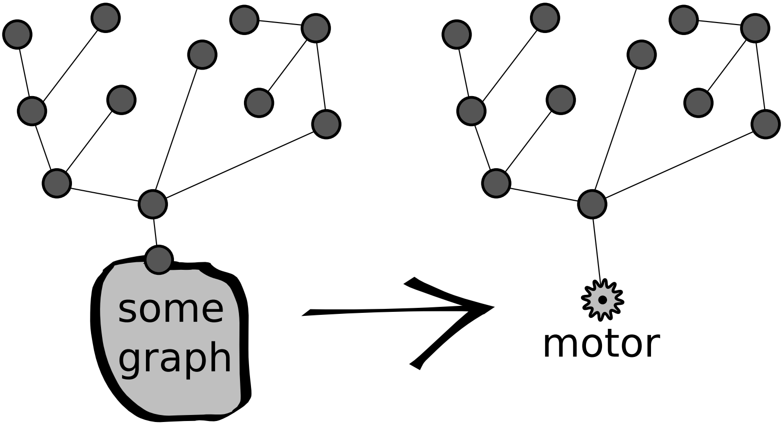

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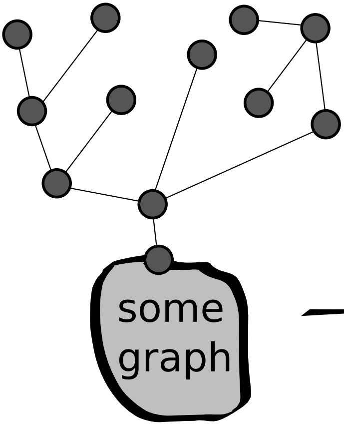

Theorem (Periodic behavior of trees with one sparse motor)

If motor m in tree graph G is sparse, then for all $v \in V(G)$ at any periodic time t, $F_v(t) = F_m(t - d)$, where d is the distance from m to v.

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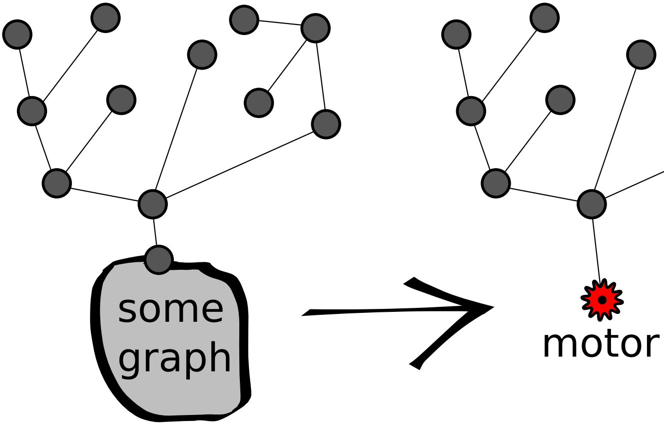

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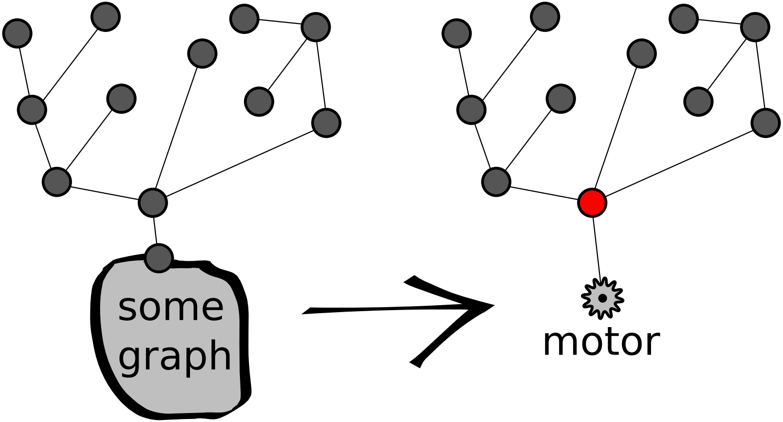

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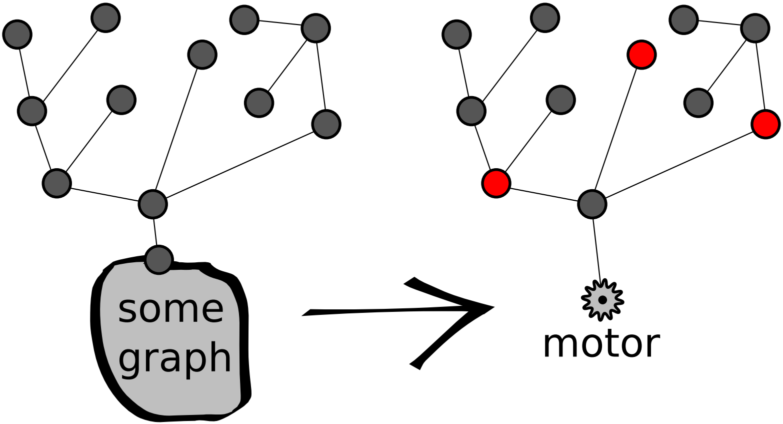

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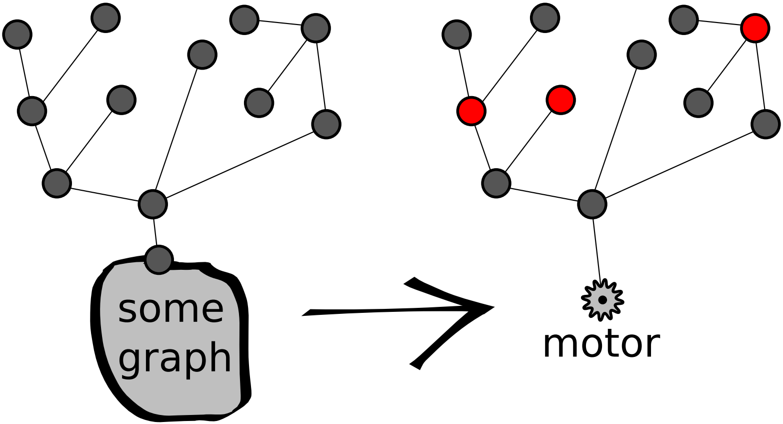

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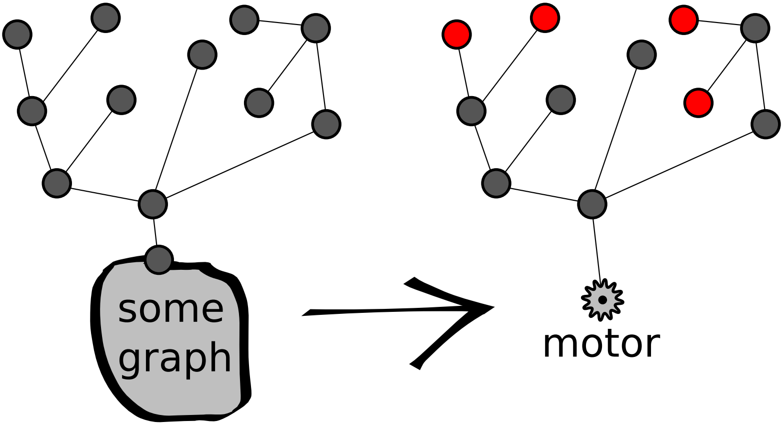

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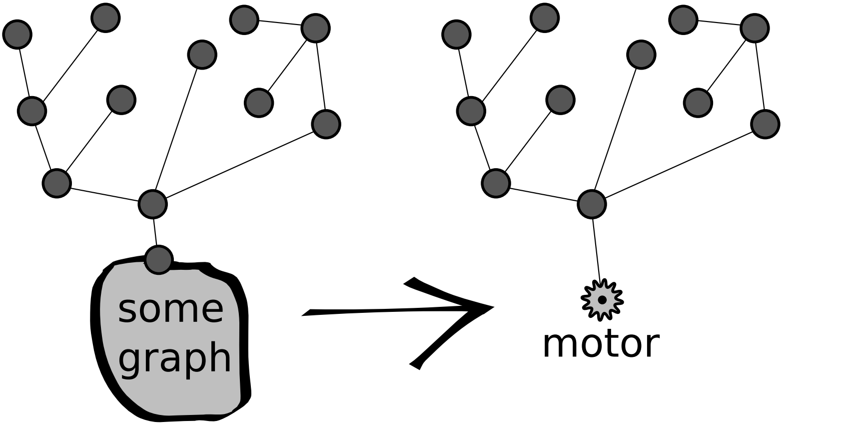

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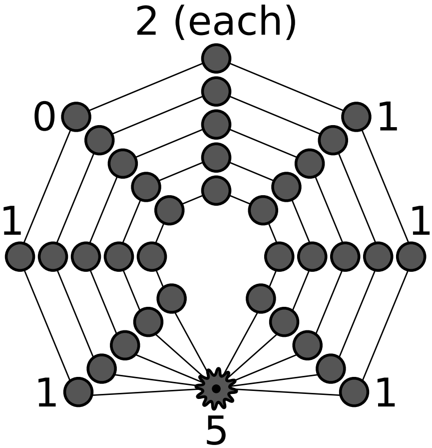
Constructing Natural Sparse and Alternating Motors



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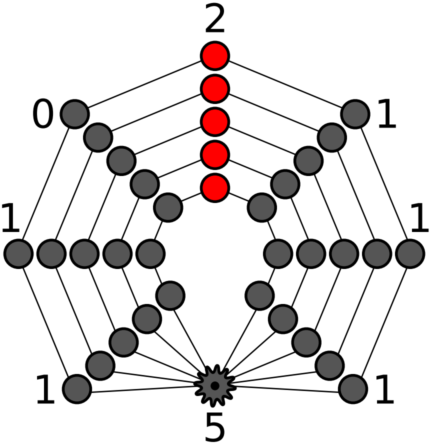

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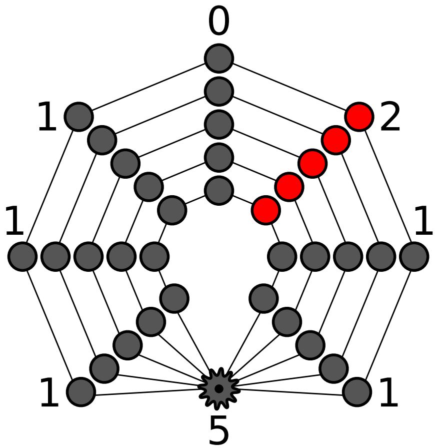

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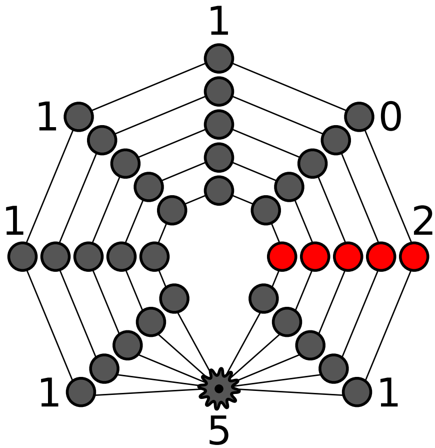

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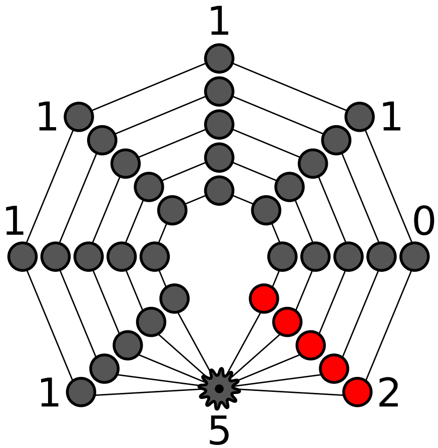

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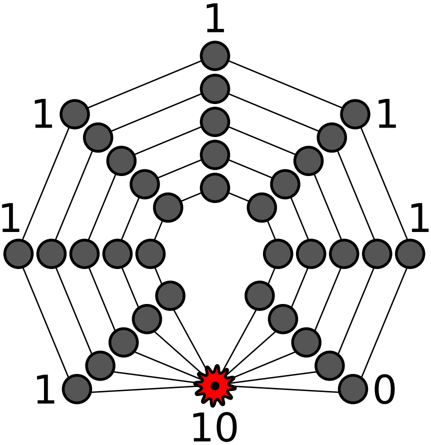

$(0, 0, 0, 0, 1,$

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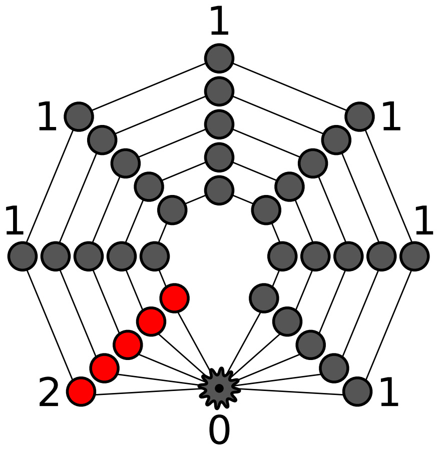

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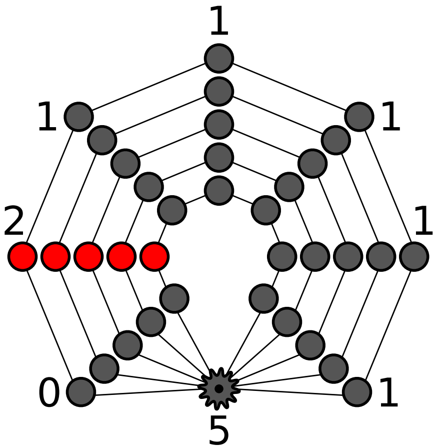

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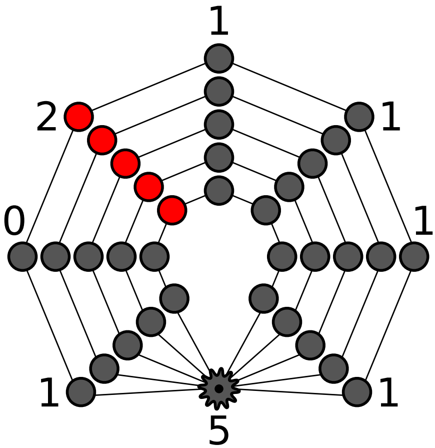

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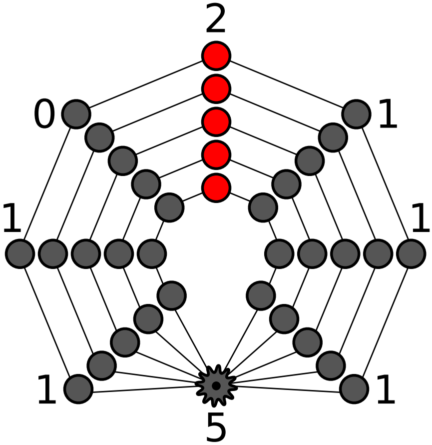

$(0, 0, 0, 0, 1, 0, 0, 0)$

Further Questions

- Can a vertex have a clumpy firing pattern in a period?
- Can every vertex firing be traced back to a "driving cycle"?
- If a graph has a possible period of length mp for some prime p, must the graph have a cycle of length np?

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