The Generalizations of the Golden Ratio: Their Powers, Continued Fractions, and Convergents

Saarik Kalia

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Abstract

The relationship between the golden ratio and continued fractions is commonly known about throughout the mathematical world: the convergents of the continued fraction are the ratios of consecutive Fibonacci numbers. The continued fractions for the powers of the golden ratio also exhibit an interesting relationship with the Lucas numbers.

In this paper, we study the silver means and introduce the bronze means, which are generalizations of the golden ratio. We correspondingly introduce the silver and bronze Fibonacci and Lucas numbers, and we prove the relationship between the convergents of the continued fractions of the powers of the silver and bronze means and the silver and bronze Fibonacci and Lucas numbers. We further generalize this to the Lucas constants, a two-parameter generalization of the golden ratio.

1 Introduction

The golden ratio exhibits an interesting relationship with continued fractions. We can summarize this relationship in three already known properties. Firstly, the n^{th} convergent of the golden ratio is $\frac{F_{n+1}}{F_n}$.[1] Secondly, the continued fraction of the n^{th} power of the golden ratio is $\{L_n; \overline{L_n}\}$ for odd n and $\{L_n - 1; \overline{1, L_n - 2}\}$ for even n.[3] Finally, the convergents of the powers of the golden ratio can be expressed as $\frac{F_{a(n+1)}}{F_{an}}$. In this paper, we will generalize the golden ratio to a group of constants and see that corresponding properties hold.

In Section 2, we will define old and new terms and list known properties. In Section 3, we will prove the three properties, which constitute the relationship between the golden ratio and continued fractions. In Section 4, we generalize this to the silver means, whose continued fractions are $\{m;\overline{m}\}$, analogous to the golden ratio's $\{1;\overline{1}\}$. To complete the generalization, not only must we generalize the Fibonacci and Lucas numbers to two families of series we will call the silver Fibonacci and Lucas numbers, but we need to also define the bronze means and the associated bronze Fibonacci and Lucas numbers. We will find that the three properties which were true of the golden ratio also hold for the silver and bronze means. In Section 5, we will go even further to a two-parameter generalization of these properties. The Lucas sequences are a family of sequences, of which the silver and bronze Fibonacci and Lucas numbers are a subset. When we define the Lucas constants, constants associated with a Lucas sequence and analogous to the silver and bronze means, we see that they obey one of the properties. The other two, however, only certain Lucas constants obey. We will discuss the nature of the continued fractions of the Lucas constants and for which Lucas constants these properties hold. Though we will find certain cases for which these properties hold, the search for an all-encompassing case will be left to further research.

2 Definitions and Properties

2.1 Old Definitions

A *continued fraction* is a form of representing a number by nested fractions, all of whose numerators are 1. For instance, the continued fraction for $\frac{9}{7}$ is $1+\cfrac{1}{3+\cfrac{1}{2}}$. The compact notation for this continued fraction is $\{1;3,2\}$. (Note a semicolon follows the first term, while commas follow the others.) The continued fraction of a number is finite if and only if that number is rational.

A *quadratic irrational* or *quadratic surd* is a number that is the solution to some quadratic equation with rational coefficients. The continued fraction of a number is *periodic*, meaning it has a repeating block, if and only if that number is a quadratic irrational. The repeating block of a periodic continued fraction is denoted by a vinculum (a horizontal line above the block).

A *reduced surd* is a quadratic surd which is greater than 1 and whose conjugate is greater than -1 and less than 0. Galois proved that the continued fraction of a number is *purely periodic*, meaning it begins with its repeating block, if and only if that number is a reduced surd. He also proved that the repeating block of a reduced surd is the mirror image of the repeating block of the negative reciprocal of its conjugate, which must also be a reduced surd.[2]

A *convergent* is the truncation of a continued fraction. For example, the second convergent of $\{3; 2, 5, 6, 8\}$ would be $\{3; 2\}$.

The *Fibonacci numbers* (F_n) are a sequence defined by the recurrence $F_{n+2} = F_{n+1} + F_n$, where $F_0 = 0$ and $F_1 = 1$.

The *Lucas numbers* (L_n) are a sequence defined by the recurrence $L_{n+2} = L_{n+1} + L_n$, where $L_0 = 2$ and $L_1 = 1$.

The *golden ratio* (ϕ) is $\frac{1+\sqrt{5}}{2}$.

The *silver means* (S_m) are $\frac{m+\sqrt{m^2+4}}{2}$. The silver means are analogues to the golden ratio, as $S_1 = \phi$.

The *Lucas sequences* are a family of sequences, consisting of two paired types of sequences. The *U-series* ($U_n(P, Q)$ or simply U_n) is defined by the recurrence $U_{n+2} = PU_{n+1} - QU_n$, where $U_0 = 0$ and $U_1 = 1$. The *V-series* ($V_n(P, Q)$ or V_n) is defined by the recurrence $V_{n+2} = PV_{n+1} - QV_n$, where $V_0 = 2$ and $V_1 = P$. The Lucas sequences are a generalization of the Fibonacci and Lucas nubmers, as $U_n(1, -1) = F_n$ and $V_n(1, -1) = L_n$.

2.2 New Definitions

The *silver Fibonacci numbers* ($F_{m,n}$) are a family of sequences defined by the recurrence $F_{m,n+2} = mF_{m,n+1} + F_{m,n}$, where $F_{m,0} = 0$ and $F_{m,1} = 1$. The silver Fibonacci numbers are a generalization of the Fibonacci numbers, as $F_{1,n} = F_n$.

The *silver Lucas numbers* ($L_{m,n}$) are a family of sequences defined by the recurrence $L_{m,n+2} = mL_{m,n+1} + L_{m,n}$, where $L_{m,0} = 2$ and $L_{m,1} = m$. The silver Lucas numbers are a generalization of the Lucas numbers, as $L_{1,n} = L_n$.

The *bronze Fibonacci numbers* ($f_{m,n}$) are a family of sequences defined by the recurrence $f_{m,n+2} = mf_{m,n+1} - f_{m,n}$, where $f_{m,0} = 0$ and $f_{m,1} = 1$.

The *bronze Lucas numbers* ($l_{m,n}$) are a family of sequence defined by the recurrence $l_{m,n+2} = ml_{m,n+1} - l_{m,n}$, where $l_{m,0} = 2$ and $l_{m,1} = m$.

The *bronze means* (B_m) are $\frac{m+\sqrt{m^2-4}}{2}$.

The *Lucas constants* ($C(P,Q)$ or C) are defined as $\frac{P+\sqrt{P^2-4Q}}{2}$. The Lucas constants are analogues to the golden ratio, as $C(1,-1) = \phi$ and generalizations of the silver and bronze means, as $C(m,-1) = S_m$ and $C(m,1) = B_m$. A Lucas constant is *degenerate* if $P^2 - 4Q$ is a perfect square.

2.3 Known Properties

These are a few known properties of the Fibonacci and Lucas numbers and the golden ratio that will be used later on.

- $L_n = F_{n+1} + F_{n-1} = 2F_{n+1} - F_n$.
- $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$.
- The continued fraction for the golden ratio is $\{1; \overline{1}\}$.
- $\phi^{n+2} = \phi^{n+1} + \phi^n$.
- $\phi^n = \frac{L_n + F_n\sqrt{5}}{2}$.
- $\phi\overline{\phi} = -1$.
- $F_n = \frac{\phi^n - \overline{\phi}^n}{\sqrt{5}}$.

3 The Golden Ratio and Continued Fractions

The relationship between the golden ratio and continued fractions can be encompassed in three properties. The first, which is commonly known, relates the convergents of the golden ratio to the Fibonacci numbers.[1] The second, which is known, but not as commonly, relates the powers of the golden ratio to the Lucas numbers.[3] The final property pertains to the convergents of the powers of the golden ratio.

3.1 The Convergents of the Golden Ratio

Theorem 3.1. *The n^{th} convergent of the golden ratio is $\frac{F_{n+1}}{F_n}$.*

Proof. We can easily prove this by induction. Clearly, this works for the case $n = 1$, as the 1^{st} convergent is 1, and $\frac{F_2}{F_1} = \frac{1}{1} = 1$. Assuming the n^{th} convergent is $\frac{F_{n+1}}{F_n}$, the $n+1^{\text{th}}$ convergent is $1 + \frac{1}{\frac{F_{n+1}}{F_n}} = 1 + \frac{F_n}{F_{n+1}} = \frac{F_{n+1}+F_n}{F_{n+1}} = \frac{F_{n+2}}{F_{n+1}}$. $\square$

3.2 The Powers of the Golden Ratio

Theorem 3.2. *The continued fraction for ϕ^n is $\{L_n; \overline{L_n}\}$ if n is odd. The continued fraction for ϕ^n is $\{L_n - 1; \overline{1, L_n - 2}\}$ if n is even.*

Proof. Let $x = \{L_n; \overline{L_n}\}$ and assume n is odd.

$$\begin{aligned}
x &= L_n + \frac{1}{x} \implies x^2 - L_n x - 1 = 0 \implies x = \frac{L_n + \sqrt{L_n^2 + 4}}{2} \\
&= \frac{L_n + \sqrt{(2F_{n+1} - F_n)^2 + 4}}{2} = \frac{L_n + \sqrt{4F_{n+1}^2 - 4F_{n+1}F_n + F_n^2 + 4}}{2} \\
&= \frac{L_n + \sqrt{4F_{n+1}F_{n-1} + F_n^2 + 4}}{2} = \frac{L_n + \sqrt{4(F_{n+1}F_{n-1} - (-1)^n) + F_n^2}}{2} \\
&= \frac{L_n + \sqrt{4F_n^2 + F_n^2}}{2} = \frac{L_n + F_n\sqrt{5}}{2} = \phi^n.
\end{aligned}$$

Now let $x = \{L_n - 1; \overline{1, L_n - 2}\}$ and assume n is even.

$$x = L_n - 1 + \cfrac{1}{1 + \cfrac{1}{x-1}} = L_n - 1 + \frac{x-1}{x} = L_n - \frac{1}{x} \implies x^2 - L_n x + 1 = 0$$

$$\begin{aligned}
&\implies x = \frac{L_n + \sqrt{L_n^2 - 4}}{2} = \frac{L_n + \sqrt{(2F_{n+1} - F_n)^2 - 4}}{2} \\
&= \frac{L_n + \sqrt{4F_{n+1}^2 - 4F_{n+1}F_n + F_n^2 - 4}}{2} = \frac{L_n + \sqrt{4F_{n+1}F_{n-1} + F_n^2 - 4}}{2} \\
&= \frac{L_n + \sqrt{4(F_{n+1}F_{n-1} - (-1)^n) + F_n^2}}{2} = \frac{L_n + \sqrt{4F_n^2 + F_n^2}}{2} = \frac{L_n + F_n\sqrt{5}}{2} \\
&= \phi^n.
\end{aligned}$$

□

3.3 The Convergents of the Powers of the Golden Ratio

Lemma 3.1. *$F_{a(n+2)} = L_a F_{a(n+1)} + F_{an}$, when a is odd.*

Proof. Let g_n follow the proposed recurrence, that is let $g_{n+2} = L_a g_{n+1} + g_n$, and $g_0 = 0$ and $g_1 = 1$. Let $G = \sum\limits_{n=0}^{\infty} g_n x^n$. Then,

$$G = L_a x G + x^2 G + x \implies G(1 - L_a x - x^2) = x \implies G = \frac{x}{1 - L_a x - x^2}.$$

By solving the denominator, we find its roots are $\frac{-L_a \pm \sqrt{L_a^2 - 4}}{2}$, which by similar methods as the one used in the first part of the previous proof, are equal to $-\overline{\phi}^a$ and $-\phi^a$.

$$\begin{aligned}
G &= -\frac{x}{(\overline{\phi}^a + x)(\phi^a + x)} = \frac{\frac{\overline{\phi}^a}{\overline{\phi}^a + x} - \frac{\phi^a}{\phi^a + x}}{F_a\sqrt{5}} = \frac{\frac{1}{1 + \frac{x}{\overline{\phi}^a}} - \frac{1}{1 + \frac{x}{\phi^a}}}{F_a\sqrt{5}} = \frac{\frac{1}{1 - \phi^a x} - \frac{1}{1 - \overline{\phi}^a x}}{F_a\sqrt{5}} \\
&\implies g_n = \frac{\phi^{an} - \overline{\phi}^{an}}{F_a\sqrt{5}} = \frac{F_{an}}{F_a}.
\end{aligned}$$

Since $\frac{F_{an}}{F_a}$ satisfies the recurrence, it follows that F_{an} would as well. □

Lemma 3.2. *$F_{a(n+2)} = L_a F_{a(n+1)} - F_{an}$, when a is even.*

Proof. Let g_n follow the proposed recurrence, that is let $g_{n+2} = L_a g_{n+1} - g_n$, and $g_0 = 0$ and $g_1 = 1$. Let $G = \sum_{n=0}^{\infty} g_n x^n$. Then,

$$G = L_a x G - x^2 G + x \implies G(1 - L_a x + x^2) = x \implies G = \frac{x}{1 - L_a x + x^2}.$$

By solving the denominator, we find its roots are $\frac{L_a \pm \sqrt{L_a^2+4}}{2}$, which are equal to $\overline{\phi}^a$ and ϕ^a.

$$G = \frac{x}{(\overline{\phi}^a - x)(\phi^a - x)} = \frac{\frac{\overline{\phi}^a}{\overline{\phi}^a - x} - \frac{\phi^a}{\phi^a - x}}{F_a\sqrt{5}} = \frac{\frac{1}{1-\frac{x}{\overline{\phi}^a}} - \frac{1}{1-\frac{x}{\phi^a}}}{F_a\sqrt{5}} = \frac{\frac{1}{1-\phi^a x} - \frac{1}{1-\overline{\phi}^a x}}{F_a\sqrt{5}}$$
$$\implies g_n = \frac{\phi^{an} - \overline{\phi}^{an}}{F_a\sqrt{5}} = \frac{F_{an}}{F_a}.$$

Since $\frac{F_{an}}{F_a}$ satisfies the recurrence, it follows that F_{an} would as well. $\square$

Theorem 3.3. *The n^{th} convergent of ϕ^a is $\frac{F_{a(n+1)}}{F_{an}}$, if a is odd, and the $2n^{th}$ convergent of ϕ^a is $\frac{F_{a(n+1)}}{F_{an}}$, if a is even.*

Proof. We can easily prove this by induction. For odd a, the case of $n = 1$ yields $\frac{F_{2a}}{F_a} = L_a$, which is the first convergent of $\phi^a = \{L_a; \overline{L_a}\}$. Assuming the n^{th} convergent is $\frac{F_{a(n+1)}}{F_{an}}$, the $n+1^{\text{th}}$ convergent is $L_a + \frac{1}{\frac{F_{a(n+1)}}{F_{an}}} = L_a + \frac{F_{an}}{F_{a(n+1)}} = \frac{L_a F_{a(n+1)} + F_{an}}{F_{a(n+1)}} = \frac{F_{a(n+2)}}{F_{a(n+1)}}$. For even a, the case of $n = 1$ yields $\frac{F_{2a}}{F_a} = L_a$, which is the second convergent of $\phi^a = \{L_a - 1; \overline{1, L_a - 2}\}$. Assuming the $2n^{\text{th}}$ convergent is $\frac{F_{a(n+1)}}{F_{an}}$, the $2n + 2^{\text{th}}$ convergent is $L_a - 1 + \cfrac{1}{1 + \cfrac{1}{\frac{F_{a(n+1)}}{F_{an}} - 1}} =$
$L_a - 1 + \cfrac{1}{1 + \frac{F_{an}}{F_{a(n+1)} - F_{an}}} = L_a - 1 + \frac{F_{a(n+1)} - F_{an}}{F_{a(n+1)}} = \frac{L_a F_{a(n+1)} - F_{a(n+1)} + F_{a(n+1)} - F_{an}}{F_{a(n+1)}} = \frac{L_a F_{a(n+1)} - F_{an}}{F_{a(n+1)}} = \frac{F_{a(n+2)}}{F_{a(n+1)}}$. $\square$

4 The Silver and Bronze Means

The properties we have found that relate the golden ratio and continued fractions can be generalized to a family of similar numbers, known as the silver means.[4] As the continued fraction of the golden ratio is $\{1;\overline{1}\}$, the continued fractions of the silver means are $\{m;\overline{m}\}$. As the Fibonacci and Lucas numbers are related to the golden ratio, the families of sequences we will call the silver Fibonacci numbers and silver Lucas numbers are related to the golden ratio similarly. To complete the generalization, we must also define another family of constants and another two families of sequences, we will call the bronze means and the bronze Fibonacci and Lucas numbers. With these terms defined, we can generalize theorems 3.1, 3.2, and 3.3.

4.1 Lemmas Pertaining to the Silver and Bronze Means

We will begin by stating lemmas analogous to the known properties listed in Section 2.3.

Lemma 4.1. $L_{m,n} = F_{m,n+1} + F_{m,n-1} = 2F_{m,n+1} - mF_{m,n}$. *Similarly,* $l_{m,n} = f_{m,n+1} - f_{m,n-1} = 2f_{m,n+1} - mf_{m,n}$.

Proof. As both sides of each equation share the same recurrence, we need only show that the cases for $n = 1$ and $n = 2$ work.

$$F_{m,2} + F_{m,0} = m + 0 = L_{m,1}.$$

$$f_{m,2} - f_{m,0} = m - 0 = l_{m,1}.$$

$$F_{m,3} + F_{m,1} = m^2 + 1 + 1 = m^2 + 2 = L_{m,2}.$$

$$f_{m,3} - f_{m,1} = m^2 - 1 - 1 = m^2 - 2 = l_{m,2}.$$

□

Lemma 4.2. $F_{m,n}^2 - F_{m,n+1}F_{m,n-1} = (-1)^{n-1}$. *Similarly,* $f_{m,n}^2 - f_{m,n+1}f_{m,n-1} = 1$.

Proof. We can prove this by induction. This works for the case $n = 1$, as $F_{m,1}^2 - F_{m,2}F_{m,0} = 1 - 0 = (-1)^0$ and $f_{m,1}^2 - f_{m,2}f_{m,0} = 1 - 0 = 1$. Assuming the property is true for n, we get

$$\begin{aligned}
F_{m,n+1}^2 - F_{m,n+2}F_{m,n} &= F_{m,n+1}^2 - F_{m,n}(mF_{m,n+1} + F_{m,n}) \\
&= F_{m,n+1}^2 - mF_{m,n+1}F_{m,n} - F_{m,n}^2 \\
&= F_{m,n+1}(F_{m,n+1} - mF_{m,n}) - F_{m,n}^2 \\
&= F_{m,n+1}F_{m,n-1} - F_{m,n}^2 = -(-1)^{n-1} = (-1)^n.
\end{aligned}$$

$$\begin{aligned}
f_{m,n+1}^2 - f_{m,n+2}f_{m,n} &= f_{m,n+1}^2 - f_{m,n}(mf_{m,n+1} - f_{m,n}) \\
&= f_{m,n+1}^2 - mf_{m,n+1}f_{m,n} + f_{m,n}^2 \\
&= f_{m,n+1}(f_{m,n+1} - mf_{m,n}) + f_{m,n}^2 \\
&= f_{m,n}^2 - f_{m,n+1}f_{m,n-1} = 1.
\end{aligned}$$

□

Lemma 4.3. *The continued fractions for the silver means are* $\{m; \overline{m}\}$. *The continued fractions for the bronze means are* $\{m-1; \overline{1, m-2}\}$.

Proof. Let $x = \{m; \overline{m}\}$.

$$x = m + \frac{1}{x} \implies x^2 - mx - 1 = 0 \implies x = \frac{m + \sqrt{m^2+4}}{2} = S_m.$$

Now let $x = \{m-1; \overline{1, m-2}\}$.

$$x = m - 1 + \frac{1}{1 + \frac{1}{x-1}} = m - 1 + \frac{x-1}{x} = m - \frac{1}{x} \implies x^2 - mx + 1 = 0$$
$$\implies x = \frac{m + \sqrt{m^2-4}}{2} = B_m.$$

□

Lemma 4.4. $S_m^{n+2} = mS_m^{n+1} + S_m^n$. *Similarly,* $B_m^{n+2} = mB_m^{n+1} - B_m^n$.

Proof. It is clear from the definition of the silver and bronze means that $S_m^2 = mS_m + 1$ and $B_m^2 = mB_m - 1$, so by multiplying the equations by S_m^n and B_m^n respectively, it follows quite simply that $S_m^{n+2} = mS_m^{n+1} + S_m^n$ and $B_m^{n+2} = mB_m^{n+1} - B_m^n$. □

Lemma 4.5. $S_m^n = \frac{L_{m,n}+F_{m,n}\sqrt{m^2+4}}{2}$. *Similarly,* $B_m^n = \frac{l_{m,n}+f_{m,n}\sqrt{m^2-4}}{2}$.

Proof. As both sides of each equation share the same recurrence, we need only show that the cases for $n = 0$ and $n = 1$ work.

$$\frac{L_{m,0} + F_{m,0}\sqrt{m^2+4}}{2} = \frac{2 + 0 \cdot \sqrt{m^2+4}}{2} = 1 = S_m^0.$$

$$\frac{L_{m,1} + F_{m,1}\sqrt{m^2+4}}{2} = \frac{m + 1 \cdot \sqrt{m^2+4}}{2} = S_m^1.$$

$$\frac{l_{m,0} + f_{m,0}\sqrt{m^2-4}}{2} = \frac{2 + 0 \cdot \sqrt{m^2-4}}{2} = 1 = B_m^0.$$

$$\frac{l_{m,1} + f_{m,1}\sqrt{m^2-4}}{2} = \frac{m + 1 \cdot \sqrt{m^2-4}}{2} = B_m^1.$$

□

Lemma 4.6. $S_m\overline{S_m} = -1$. *Similarly,* $B_m\overline{B_m} = 1$.

Proof.

$$S_m\overline{S_m} = \frac{m + \sqrt{m^2+4}}{2} \cdot \frac{m - \sqrt{m^2+4}}{2} = \frac{m^2 - m^2 - 4}{4} = -1.$$

$$B_m\overline{B_m} = \frac{m + \sqrt{m^2-4}}{2} \cdot \frac{m - \sqrt{m^2-4}}{2} = \frac{m^2 - m^2 + 4}{4} = 1.$$

□

Lemma 4.7. $F_{m,n} = \frac{S_m^n - \overline{S_m}^n}{\sqrt{m^2+4}}$. *Similarly,* $f_{m,n} = \frac{B_m^n - \overline{B_m}^n}{\sqrt{m^2-4}}$.

Proof.

$$\frac{S_m^n - \overline{S_m}^n}{\sqrt{m^2+4}} = \frac{\frac{L_{m,n}+F_{m,n}\sqrt{m^2+4}}{2} - \frac{L_{m,n}-F_{m,n}\sqrt{m^2+4}}{2}}{\sqrt{m^2+4}} = \frac{F_{m,n}\sqrt{m^2+4}}{\sqrt{m^2+4}} = F_{m,n}.$$

$$\frac{B_m^n - \overline{B_m}^n}{\sqrt{m^2-4}} = \frac{\frac{l_{m,n}+f_{m,n}\sqrt{m^2-4}}{2} - \frac{l_{m,n}-f_{m,n}\sqrt{m^2-4}}{2}}{\sqrt{m^2-4}} = \frac{f_{m,n}\sqrt{m^2-4}}{\sqrt{m^2-4}} = f_{m,n}.$$

□

4.2 The Convergents of the Silver and Bronze Means

Theorem 4.1. *The n^{th} convergents of the silver means are $\frac{F_{m,n+1}}{F_{m,n}}$ and the $2n^{th}$ convergents of the bronze means are $\frac{f_{m,n+1}}{f_{m,n}}$.*

Proof. We can easily prove this by induction. Clearly, this works for the case $n = 1$, the 1^{st} convergents of the silver means are m and the 2^{nd} convergents of the silver means are also $m-1+\frac{1}{1} = m$, and $\frac{F_{m,2}}{F_{m,1}} = \frac{f_{m,2}}{f_{m,1}} = \frac{m}{1} = m$. Assuming the theorem, the $n+1^{\text{th}}$ convergents of the silver means are $m + \frac{1}{\frac{F_{m,n+1}}{F_{m,n}}} = m+\frac{F_{m,n}}{F_{m,n+1}} = \frac{mF_{m,n+1}+F_{m,n}}{F_{m,n+1}} = \frac{F_{m,n+2}}{F_{m,n+1}}$, and the $n+1^{\text{th}}$ convergents of the bronze means are $m-1+\cfrac{1}{1+\cfrac{1}{\frac{f_{m,n+1}}{f_{m,n}}-1}} = m-1+\cfrac{1}{\frac{f_{m,n+1}}{f_{m,n+1}-f_{m,n}}} = m-1+\frac{f_{m,n+1}-f_{m,n}}{f_{m,n+1}} = \frac{mf_{m,n+1}-f_{m,n+1}+f_{m,n+1}-f_{m,n}}{f_{m,n+1}} = \frac{f_{m,n+2}}{f_{m,n+1}}$. $\square$

4.3 The Powers of the Silver and Bronze Means

Theorem 4.2. *$S_m^n = S_{L_{m,n}}$, if n is odd, and $S_m^n = B_{L_{m,n}}$, if n is even. In addition, $B_m^n = B_{l_{m,n}}$.*

Proof. $S_m^n = \frac{L_{m,n}+F_{m,n}\sqrt{m^2+4}}{2} = \frac{L_{m,n}+\sqrt{m^2F_{m,n}^2+4F_{m,n}^2}}{2}$. If n is odd,

$$\begin{aligned}
S_m^n &= \frac{L_{m,n} + \sqrt{m^2F_{m,n}^2 + 4F_{m,n+1}F_{m,n-1} + 4}}{2} \\
&= \frac{L_{m,n} + \sqrt{m^2F_{m,n}^2 + 4F_{m,n+1}^2 - 4mF_{m,n+1}F_{m,n} + 4}}{2} \\
&= \frac{L_{m,n} + \sqrt{(2F_{m,n+1} - mF_{m,n})^2 + 4}}{2} = \frac{L_{m,n} + \sqrt{L_{m,n}^2 + 4}}{2} = S_{L_{m,n}}.
\end{aligned}$$

If n is even,

$$\begin{aligned}
S_m^n &= \frac{L_{m,n} + \sqrt{m^2F_{m,n}^2 + 4F_{m,n+1}F_{m,n-1} - 4}}{2} \\
&= \frac{L_{m,n} + \sqrt{m^2F_{m,n}^2 + 4F_{m,n+1}^2 - 4mF_{m,n+1}F_{m,n} - 4}}{2} \\
&= \frac{L_{m,n} + \sqrt{(2F_{m,n+1} - mF_{m,n})^2 - 4}}{2} = \frac{L_{m,n} + \sqrt{L_{m,n}^2 - 4}}{2} = B_{L_{m,n}}.
\end{aligned}$$

For any n,

$$\begin{aligned}
B_m^n &= \frac{l_{m,n} + f_{m,n}\sqrt{m^2-4}}{2} = \frac{l_{m,n} + \sqrt{m^2 f_{m,n}^2 - 4 f_{m,n}^2}}{2} \\
&= \frac{l_{m,n} + \sqrt{m^2 f_{m,n}^2 - 4 f_{m,n+1} f_{m,n-1} - 4}}{2} \\
&= \frac{l_{m,n} + \sqrt{m^2 f_{m,n}^2 + 4 f_{m,n+1}^2 - 4m f_{m,n+1} f_{m,n} - 4}}{2} \\
&= \frac{l_{m,n} + \sqrt{(2f_{m,n+1} - m f_{m,n})^2 - 4}}{2} = \frac{l_{m,n} + \sqrt{l_{m,n}^2 - 4}}{2} = B_{l_{m,n}}.
\end{aligned}$$

□

4.4 The Convergents of the Powers of the Silver and Bronze Means

Lemma 4.8. *$F_{L_{m,a},n} = \frac{F_{m,an}}{F_{m,a}}$, if a is odd.*

Proof. Assume a is odd and let $G = \sum_{n=0}^{\infty} F_{L_{m,a},n} x^n$. Since $F_{L_{m,a},n+2} = L_{m,a} F_{L_{m,a},n+1} + F_{L_{m,a},n}, F_{L_{m,a},0} = 0, F_{L_{m,a},1} = 1$,

$$\begin{aligned}
&G = L_{m,a} x G + x^2 G + x \implies G(1 - L_{m,a} x - x^2) = x \\
&\implies G = \frac{x}{1 - L_{m,a} x - x^2} = -\frac{x}{(\overline{S_m}^a + x)(S_m^a + x)} = \frac{\frac{\overline{S_m}^a}{\overline{S_m}^a + x} - \frac{S_m^a}{S_m^a + x}}{F_{m,a}\sqrt{m^2+4}} \\
&= \frac{\frac{1}{1+\frac{x}{\overline{S_m}^a}} - \frac{1}{1+\frac{x}{S_m^a}}}{F_{m,a}\sqrt{m^2+4}} = \frac{\frac{1}{1-S_m^a x} - \frac{1}{1-\overline{S_m}^a x}}{F_{m,a}\sqrt{m^2+4}} \implies F_{L_{m,a},n} = \frac{S_m^{an} - \overline{S_m}^{an}}{F_{m,a}\sqrt{m^2+4}} \\
&= \frac{F_{m,an}}{F_{m,a}}.
\end{aligned}$$

□

Lemma 4.9. *$f_{L_{m,a},n} = \frac{F_{m,an}}{F_{m,a}}$, if a is even.*

Proof. Assume a is even and let $G = \sum_{n=0}^{\infty} f_{L_{m,a},n} x^n$. Since $f_{L_{m,a},n+2} =$

$L_{m,a}f_{L_{m,a},n+1} - f_{L_{m,a},n}, f_{L_{m,a},0} = 0, f_{L_{m,a},1} = 1,$

$$G = L_{m,a}xG - x^2G + x \implies G(1 - L_{m,a}x + x^2) = x$$

$$\implies G = \frac{x}{1 - L_{m,a}x + x^2} = \frac{x}{(\overline{S_m}^a - x)(S_m^a - x)} = \frac{\frac{\overline{S_m}^a}{\overline{S_m}^a - x} - \frac{S_m^a}{S_m^a - x}}{F_{m,a}\sqrt{m^2+4}}$$

$$= \frac{\frac{1}{1-\frac{x}{\overline{S_m}^a}} - \frac{1}{1-\frac{x}{S_m^a}}}{F_{m,a}\sqrt{m^2+4}} = \frac{\frac{1}{1-S_m^a x} - \frac{1}{1-\overline{S_m}^a x}}{F_{m,a}\sqrt{m^2+4}} \implies f_{L_{m,a},n} = \frac{S_m^{an} - \overline{S_m}^{an}}{F_{m,a}\sqrt{m^2+4}}$$

$$= \frac{F_{m,an}}{F_{m,a}}.$$

□

Lemma 4.10. $f_{l_{m,a},n} = \frac{f_{m,an}}{f_{m,a}}$.

Proof. Let $G = \sum_{n=0}^{\infty} f_{l_{m,a},n}x^n$. Since $f_{l_{m,a},n+2} = l_{m,a}f_{l_{m,a},n+1} - f_{l_{m,a},n}, f_{l_{m,a},0} = 0, f_{l_{m,a},1} = 1,$

$$G = l_{m,a}xG - x^2G + x \implies G(1 - l_{m,a}x + x^2) = x$$

$$\implies G = \frac{x}{1 - l_{m,a}x + x^2} = \frac{x}{(\overline{B_m}^a - x)(B_m^a - x)} = \frac{\frac{\overline{B_m}^a}{\overline{B_m}^a - x} - \frac{B_m^a}{B_m^a - x}}{f_{m,a}\sqrt{m^2-4}}$$

$$= \frac{\frac{1}{1-\frac{x}{\overline{B_m}^a}} - \frac{1}{1-\frac{x}{B_m^a}}}{f_{m,a}\sqrt{m^2-4}} = \frac{\frac{1}{1-B_m^a x} - \frac{1}{1-\overline{B_m}^a x}}{f_{m,a}\sqrt{m^2-4}} \implies f_{l_{m,a},n} = \frac{B_m^{an} - \overline{B_m}^{an}}{f_{m,a}\sqrt{m^2-4}}$$

$$= \frac{f_{m,an}}{f_{m,a}}.$$

□

Theorem 4.3. *The convergents of S_m^a are $\frac{F_{m,a(n+1)}}{F_{m,an}}$ and the convergents of B_m^a are $\frac{f_{m,a(n+1)}}{f_{m,an}}$.*

Proof. By theorems 4.1 and 4.2, the convergents of $S_m^a = S_{L_{m,a}}$ are $\frac{F_{L_{m,a},n+1}}{F_{L_{m,a},n}}$ if a is odd, the convergents of $S_m^a = B_{L_{m,a}}$ are $\frac{f_{L_{m,a},n+1}}{f_{L_{m,a},n}}$ if a is even, and the convergents of $B_m^a = B_{l_{m,a}}$ are $\frac{f_{l_{m,a},n+1}}{f_{l_{m,a},n}}$. Thus by the lemmas 4.8, 4.9, 4.10, the convergents of S_m^a must be $\frac{F_{m,a(n+1)}}{F_{m,an}}$, and the convergents of B_m^a must be $\frac{f_{m,a(n+1)}}{f_{m,an}}$. □

5 The Lucas Constants

Our properties about the silver and bronze means can be even further generalized to a family of sequences known as the Lucas sequences. While the silver Fibonacci numbers generalize the coefficient of the first term in the recurrence, the Lucas sequences also generalize the coefficient of the second term. We need to also define a set of constants analogous to the golden ratio, silver means, and bronze means, which we will call the Lucas constants. Our properties will relate to the Lucas constants and their associated Lucas sequences. The second property holds for the Lucas constants, however the first does not hold for all Lucas sequences, and we will therefore discuss it last. As a result, while the underlying theorem behind the third property does hold, the actual property only holds in some cases.

5.1 Lemmas Pertaining to the Lucas Constants

Again we will begin by stating lemmas analogous to the known properties listed in Section 2.3.

Lemma 5.1. $V_n = U_{n+1} - QU_{n-1} = 2U_{n+1} - PU_n$.

Proof. As both sides of the equation share the same recurrence, we need only show that the cases for $n = 1$ and $n = 2$ work.

$$U_2 - QU_0 = P = V_0.$$

$$U_3 - QU_1 = P^2 - Q - Q = P^2 - 2Q = V_2.$$

□

Lemma 5.2. $U_n^2 - U_{n+1}U_{n-1} = Q^{n-1}$.

Proof. We can prove this by induction. This works for the case $n = 1$, as $U_1^2 - U_2U_0 = 1 - 0 = 1 = Q^0$. Assuming the property is true for n, we get

$$\begin{aligned} U_{n+1}^2 - U_{n+2}U_n &= U_{n+1}^2 - U_n(PU_{n+1} - QU_n) = U_{n+1}^2 - PU_{n+1}U_n + QU_n^2 \\ &= U_{n+1}(U_{n+1} - PU_n) + QU_n^2 = QU_n^2 - QU_{n+1}U_{n-1} = Q^n. \end{aligned}$$

□

Lemma 5.3. $C^{n+2} = PC^{n+1} - QC^n$.

Proof. It is clear from the definition of the Lucas constants that $C^2 = PC - Q$, so by multiplying the equations by C^n respectively, it follows quite simply that $C^{n+2} = PC^{n+1} - QC^n$. $\square$

Lemma 5.4. $C^n = \frac{V_n + U_n\sqrt{P^2-4Q}}{2}$.

Proof. As both sides of the equation share the same recurrence, we need only show that the cases for $n = 0$ and $n = 1$ work.

$$\frac{V_0 + U_0\sqrt{P^2-4Q}}{2} = \frac{2}{2} = 1 = C^0.$$

$$\frac{V_1 + U_1\sqrt{P^2-4Q}}{2} = \frac{P + \sqrt{P^2-4Q}}{2} = C^1.$$

$\square$

Lemma 5.5. $C\overline{C} = Q$.

Proof.

$$C\overline{C} = \frac{P + \sqrt{P^2-4Q}}{2} \cdot \frac{P - \sqrt{P^2-4Q}}{2} = \frac{P^2 - P^2 + 4Q}{4} = Q.$$

$\square$

Lemma 5.6. $U_n = \frac{C^n - \overline{C}^n}{\sqrt{P^2-4Q}}$.

Proof.

$$\frac{C^n - \overline{C}^n}{\sqrt{P^2-4Q}} = \frac{\frac{V_n+U_n\sqrt{P^2-4Q}}{2} - \frac{V_n-U_n\sqrt{P^2-4Q}}{2}}{\sqrt{P^2-4Q}} = \frac{U_n\sqrt{P^2-4Q}}{\sqrt{P^2-4Q}} = U_n.$$

$\square$

5.2 The Powers of the Lucas Constants

Theorem 5.1. $C^n = C(V_n, Q^n)$.

Proof.

$$
\begin{aligned}
C^n &= \frac{V_n + U_n\sqrt{P^2-4Q}}{2} = \frac{V_n + \sqrt{P^2U_n^2 - 4QU_n^2}}{2} \\
&= \frac{V_n + \sqrt{P^2U_n^2 - 4QU_{n+1}U_{n-1} - 4Q^n}}{2} \\
&= \frac{V_n + \sqrt{P^2U_n^2 + 4U_{n+1}^2 - 4PU_{n+1}U_n - 4Q^n}}{2} \\
&= \frac{V_n + \sqrt{(2U_{n+1} - PU_n)^2 - 4Q^n}}{2} = \frac{V_n + \sqrt{V_n^2 - 4Q^n}}{2} = C(V_n, Q^n).
\end{aligned}
$$

□

5.3 The Convergents of the Powers of the Lucas Constants

The underlying property, which we have so far called a lemma (lemmas 3.1 and 3.2 for the golden ratio and lemmas 4.8, 4.9, and 4.10 for the silver and bronze means) holds for the Lucas constants, though what we have thus far called a theorem (theorem 3.3 for the golden ratio and theorem 4.3 for the silver and bronze means) only holds for some values, so we will call them a theorem and corollary respectively in this section.

Theorem 5.2. $U_n(V_a, Q^a) = \frac{U_{an}}{U_a}$.

Proof. Let $G = \sum_{n=0}^{\infty} U_n(V_a, Q^a)x^n$. Since $U_{n+2}(V_a, Q^a) = V_aU_{n+1}(V_a, Q^a) - Q^aU_n(V_a, Q^a), U_0(V_a, Q^a) = 0, U_1(V_a, Q^a) = 1$,

$$
\begin{aligned}
G = V_axG - Q^ax^2G + x &\implies G(1 - V_ax + Q^ax^2) = x \\
&\implies G = \frac{x}{1 - V_ax + Q^ax^2} = \frac{x}{(1 - C^ax)(1 - \overline{C}^a x)} = \frac{\frac{1}{1-C^ax} - \frac{1}{1-\overline{C}^a x}}{U_a\sqrt{P^2-4Q}} \\
&\implies U_n(V_a, Q^a) = \frac{C^{an} - \overline{C}^{an}}{U_a\sqrt{P^2-4Q}} = \frac{U_{an}}{U_a}.
\end{aligned}
$$

□

Corollary 5.1. *If the convergents of $C(V_a, Q^a)$ are $\frac{U_{n+1}(V_a,Q^a)}{U_n(V_a,Q^a)}$ (we will discuss in the next subsection for which (P,Q) pairs the convergents of C are $\frac{U_{n+1}}{U_n}$), then the n^{th} convergent of C^a is $\frac{U_{a(n+1)}}{U_{an}}$.*

Proof. Since $C^a = C(V_a, Q^a)$, and $\frac{U_{n+1}(V_a,Q^a)}{U_n(V_a,Q^a)} = \frac{\frac{U_{a(n+1)}}{U_a}}{\frac{U_{an}}{U_a}} = \frac{U_{a(n+1)}}{U_{an}}$, the corollary follows quite simply. $\square$

5.4 The Convergents of the Lucas Constants

The generalization of this property would be that the convergents of C are $\frac{U_{n+1}}{U_n}$. This, however, does not hold for all (P, Q) pairs, though it does hold for some. To find out which it does hold for, we must first analyze the nature of the continued fractions of quadratic surds, and more specifically the Lucas constants.

Theorem 5.3. *Let $\frac{A+\sqrt{D}}{b}$ be a quadratic surd, where A, b, and D are integers, and let a be the remainder when A is divided by b. Then if $a^2 < D$ and $(b-a)^2 < D$, the repeating block of the continued fraction begins with the first or second term.*

Proof. If our quadratic surd added to or subtracted by some integer is a reduced surd (which is purely periodic), then it follows quite simply that our quadratic surd's repeating block must start with the second term. In other words, we must show that there exists some integer k such that $\frac{A+bk+\sqrt{D}}{b} > 1$ and $-1 < \frac{A+bk-\sqrt{D}}{b} < 0$, given $a^2 < D$ and $(b-a)^2 < D$. Now by manipulating our second given equality, we get

$$(b-a)^2 < D \implies b - a < \sqrt{D} \implies b < a + \sqrt{D} \implies 1 < \frac{a+\sqrt{D}}{b}.$$

As $A + bk \geq a$, our first desired equality is satisfied for any k. Now by manipulating the second desired equality, we get

$$\begin{aligned} -1 < \frac{A+bk-\sqrt{D}}{b} < 0 &\implies 1 > \frac{\sqrt{D}-(A+bk)}{b} > 0 \\ &\implies b > \sqrt{D} - (A+bk) > 0 \\ &\implies A + b(k+1) > \sqrt{D} > A + bk \\ &\implies (A + b(k+1))^2 > D > (A+bk)^2. \end{aligned}$$

Now we can clearly see that for any value of D (keeping in mind that D will never be a perfect square), there exists some k for which the above equality is true, as long as D is greater than the least possible value of $(A+bk)^2$, which is a^2. $\square$

Corollary 5.2. *The repeating block for the continued fraction of a non-degenerate Lucas constant begins with the first or second term.*

Proof. Since in the case of the Lucas constants $b = 2$, the only possible values for a are 0, when P is even and 1, when P is odd. If P is odd, then $D = P^2 - 4Q$ must also be odd. Then, excluding the degenerate value of $D = 1$, both equalities are easily satisfied. If P is even, then $P^2 \equiv 0 \pmod 4$. D therefore must also be, so again excluding the degenerate value of $D = 0$, the equalities are satisfied. $\square$

Theorem 5.4. *Let $\frac{A_1+\sqrt{D}}{b}$ and $\frac{A_2+\sqrt{D}}{b}$ be quadratic surds whose repeating blocks begin with the second term. If $A_1 + A_2$ is divisible by b, then the repeating blocks of the continued fractions of the two quadratic surds are mirror images of each other, excluding the last term in each of them.*

Proof. Since the repeating block of $\frac{A_1+\sqrt{D}}{b}$ begins with the second term, then there exists some reduced surd $\frac{a+\sqrt{D}}{b}$, for which our quadratic surd may be expressed as $\frac{a+bk_1+\sqrt{D}}{b}$, where k_1 is an integer. We know that the repeating blocks of $\frac{a+\sqrt{D}}{b}$ and the opposite reciprocal of its conjugate are mirror images of each other, so the repeating blocks of $\frac{a+\sqrt{D}}{b}$ and $\frac{\sqrt{D}-a}{b}$ must also be mirror images of each other, though the latter is shifted by a term. If $A_1 + A_2 = a + bk_1 + A_2$ is divisible by b, then A_2 must be expressible as $bk_2 - a$, where k_2 is an integer. Thus, $\frac{A_2+\sqrt{D}}{b}$ differs from $\frac{\sqrt{D}-a}{b}$ only in the first term, as $\frac{A_1+\sqrt{D}}{b}$ differs from $\frac{a+\sqrt{D}}{b}$ only in the first term as well. In the latter example, however, it changes the first term in the repeating block, making it not the same as the last term in the repeating block in the other. Therefore, if $\frac{A_1+\sqrt{D}}{b}$ and $\frac{A_2+\sqrt{D}}{b}$ have repeating blocks beginning with the second term, their repeating blocks are mirror images of each other, excluding the last term in each of them. $\square$

Corollary 5.3. *With the exclusion of the last term, the repeating block of the continued fraction of a Lucas constant is palindromic.*

Proof. Since $P + P = 2P$ must be divisible by 2, apart from the last term, the repeating block of a Lucas constant is the mirror image of itself, or, in other words, palindromic. $\square$

Now that we know the nature of the continued fractions of the Lucas constants, we can explore which (P, Q) pairs, the property applies for. Through the use of a program, we can find various pairs which work. It is clear that for $Q < 0$, the property only holds when P is divisible by $-Q$. For $Q > 0$, the property holds when P is divisible by Q but also in other more obscure cases. Here, we will prove these two cases for which the property holds.

Theorem 5.5. *If $P > 0$, $Q < 0$, and P is divisible by $-Q$, the n^{th} convergent of C is $\frac{U_{n+1}}{U_n}$.*

Proof. First we must show that the continued fraction for C is $\{P; \overline{-\frac{P}{Q}, P}\}$. Let $x = \{P; \overline{-\frac{P}{Q}, P}\}$. Then,

$$\begin{aligned} x = P + \frac{1}{-\frac{P}{Q} + \frac{1}{x}} = P - \frac{Qx}{Px - Q} &\implies Px^2 - Qx = P^2x - PQ - Qx \\ &\implies Px^2 - P^2x + PQ = 0 \implies x^2 - Px + Q = 0 \\ &\implies x = \frac{P + \sqrt{P^2 - 4Q}}{2} = C. \end{aligned}$$

Now we can prove the theorem by induction. It holds for $n = 1$, since the 1st convergent is P and $\frac{U_2}{U_1} = \frac{P}{1} = P$, and $n = 2$ since the 2nd convergent is $P + \frac{1}{-\frac{P}{Q}} = P - \frac{Q}{P} = \frac{P^2-Q}{P}$ and $\frac{U_3}{U_2} = \frac{P^2-Q}{P}$. For the case of $n + 2$, we get

$$\begin{aligned} P + \frac{1}{-\dfrac{P}{Q} + \dfrac{1}{\frac{U_{n+1}}{U_n}}} = P + \frac{1}{-\frac{P}{Q} + \frac{U_n}{U_{n+1}}} &= P + \frac{QU_{n+1}}{QU_n - PU_{n+1}} \\ &= \frac{PQU_n - P^2U_{n+1} + QU_{n+1}}{QU_n - PU_{n+1}} = \frac{P^2U_{n+1} - PQU_n - QU_{n+1}}{PU_{n+1} - QU_n} \\ &= \frac{PU_{n+2} - QU_{n+1}}{U_{n+2}} = \frac{U_{n+3}}{U_{n+2}}. \end{aligned}$$

□

Theorem 5.6. *If $P > 0$, $Q > 0$, and P is divisible by Q, the $2n^{th}$ convergent of C is $\frac{U_{n+1}}{U_n}$.*

Proof. Again, we must show that the continued fraction for C is $\{P-1;\overline{1,\frac{P}{Q}-2,1,P-2}\}$. Let $x=\{P-1;\overline{1,\frac{P}{Q}-2,1,P-2}\}$. Then,

$$x = P-1+\cfrac{1}{1+\cfrac{1}{\cfrac{P}{Q}-2+\cfrac{1}{1+\cfrac{1}{x-1}}}} = P-1+\cfrac{1}{1+\cfrac{1}{\cfrac{P}{Q}-2+\cfrac{1}{\cfrac{x}{x-1}}}}$$

$$= P-1+\cfrac{1}{1+\cfrac{1}{\cfrac{P}{Q}-2+\cfrac{x-1}{x}}} = P-1+\cfrac{1}{1+\cfrac{Qx}{Px-2Qx+Qx-Q}}$$

$$= P-1+\cfrac{1}{1+\cfrac{Qx}{Px-Qx-Q}} = P-1+\frac{Px-Qx-Q}{Px-Qx-Q+Qx}$$

$$= P-1+\frac{Px-Qx-Q}{Px-Q} = P-\frac{Qx}{Px-Q} = \frac{P^2x-PQ-Qx}{Px-Q}$$
$$\implies Px^2-Qx = P^2x-PQ-Qx \implies Px^2-P^2x+PQ=0$$
$$\implies x^2-Px+Q=0 \implies x=C.$$

Now we can prove the theorem by induction. It works for the case of $n=1$ because $P-1+\frac{1}{1}=P=\frac{P}{1}=\frac{U_2}{U_1}$ and for the case of $n=2$ because $P-1+\cfrac{1}{1+\cfrac{1}{\frac{P}{Q}-2+\frac{1}{1}}} = P-1+\cfrac{1}{1+\cfrac{1}{\frac{P}{Q}-1}} = P-1+\cfrac{1}{1+\cfrac{Q}{P-Q}} =$

$P - 1 + \dfrac{P-Q}{P} = P - \frac{Q}{P} = \frac{P^2-Q}{P} = \frac{U_3}{U_2}$. For the case of $n+2$, we get

$$P - 1 + \cfrac{1}{1 + \cfrac{1}{\dfrac{P}{Q} - 2 + \cfrac{1}{1 + \cfrac{1}{\dfrac{U_{n+1}}{U_n} - 1}}}} = \frac{P^2\frac{U_{n+1}}{U_n} - PQ - Q\frac{U_{n+1}}{U_n}}{P\frac{U_{n+1}}{U_n} - Q}$$

$$= \frac{P^2U_{n+1} - PQU_n - QU_{n+1}}{PU_{n+1} - QU_n}$$

$$= \frac{PU_{n+2} - QU_{n+1}}{U_{n+2}} = \frac{U_{n+3}}{U_{n+2}}.$$

$\square$

Two more obscure cases for which this property holds are: when P can be expressed as $a^2(n+1)+a$ and $Q = a^2$, in which case the continued fraction is of the form $\{a^2(n+1)+a-1; \overline{1, n, a, n, 1, a^2(n+1)+a-2}\}$; and when P can be expressed as $2a(2n+3)$ and $Q = 4a$, in which case the continued fraction is $\{2a(2n+3)-1; \overline{1, n, 2, a(n+1)+\frac{a-2}{2}, 2, n, 1, 2a(2n+3)-2}\}$.

6 Further Research

The most obvious topic of further research would be finding Lucas constants for which the convergents are $\frac{U_{n+1}}{U_n}$. It seems that for $Q < 0$, the only case in which the property holds is when P is divisible by $-Q$, and this would be a useful fact to prove. Then, finding other cases where $Q > 0$, for which this property holds, and even proving that certain cases can cover all the Lucas constants for which this holds, would be an area of interest. Another interesting conjecture to prove or disprove would be that for all cases where $Q < 0$, the n^{th} convergent is $\frac{U_{n+1}}{U_n}$, while where $Q > 0$, the $2n^{\text{th}}$ covergent is $\frac{U_{n+1}}{U_n}$.

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