Distinguishing Knots: The Jones Polynomial and Integer Invariants

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Reading Group

Yulia's Dream

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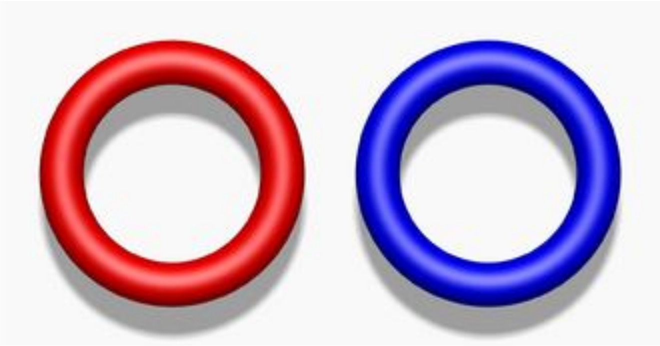
What are knots?

- Knots can be found in everyday life, such as ropes, shoe laces, and DNA molecules.
- A **mathematical knot** is defined as a closed, non-self-intersecting curve that is in three dimensions.

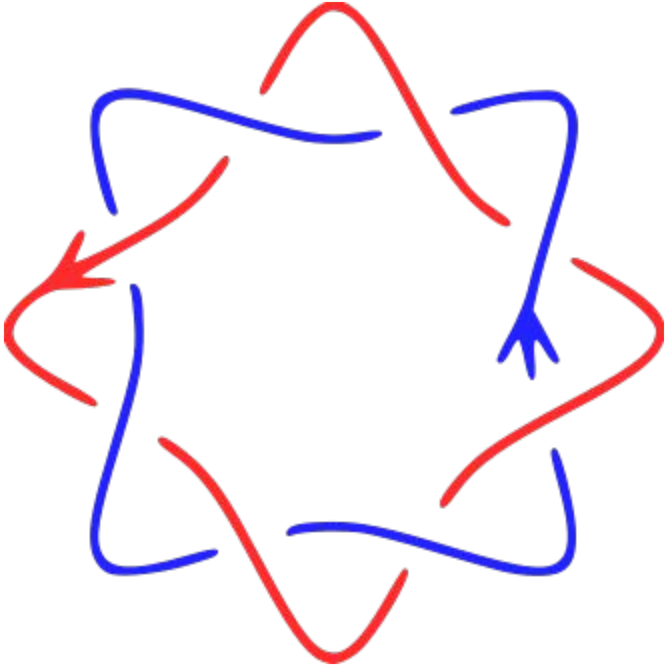
Creating the Trefoil knot

What is a link?

- A collection of one or more knots that are are considered together is called a **link**.
- A link which is equivalent (under ambient isotopy) to finitely many disjoint circles in the plane called an **unlink**.



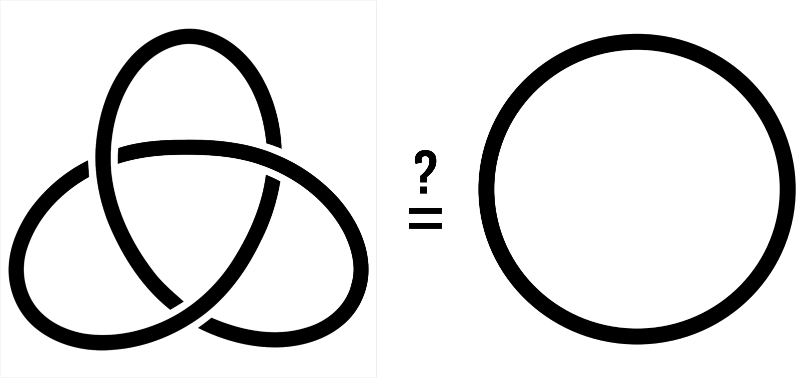
An unlink of 2 components



A 2-component link

The Core Problem in Knot Theory

- The core problem in knot theory is **determining whether two links are the same.**
- By definition, two links are the same if one can be transformed into the other without cutting it. Such a transformation is an example of an **ambient isotopy**.

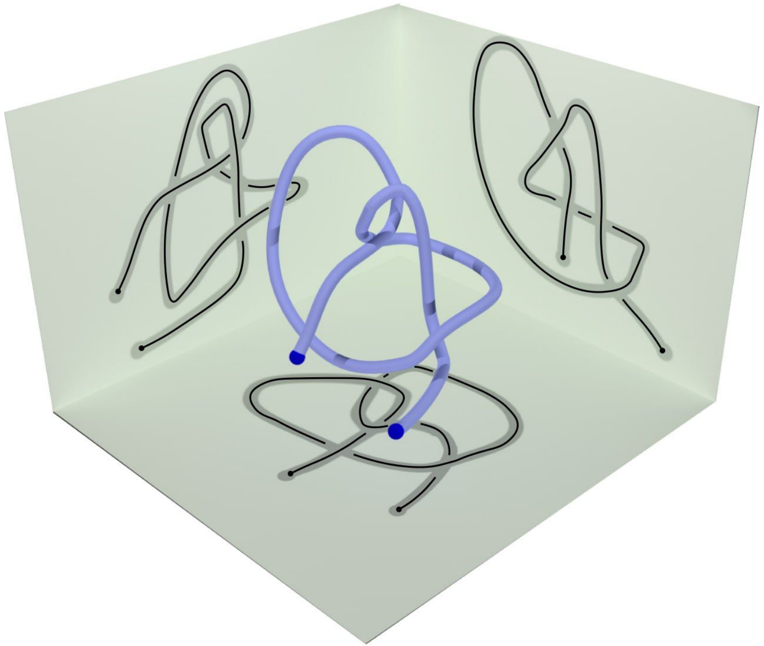


Is the trefoil knot equivalent to the unknot?

The Reidemeister moves

- The pictures of knots before are called **knot projections**. This is a regular projection to some plane in space, in which no three points on the knot project to the same point.

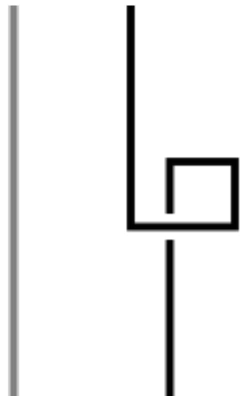
- The places where the knot crosses itself in the 2D picture are called **crossings** of the projection.



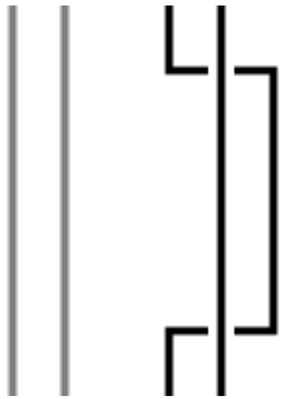
3 different knot projections

The Reidemeister moves

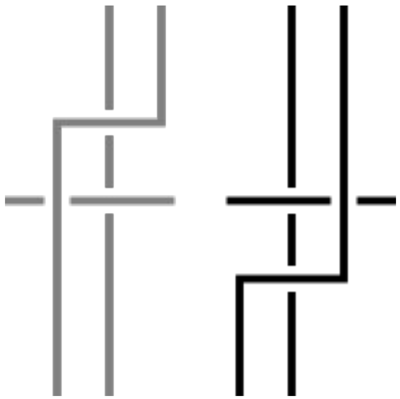
Theorem (Reidemeister)
Two links can be smoothly transformed into each other if and only if any projection of one link can be transformed into a projection of the other link through a series of Reidemeister moves.



Type 1 move



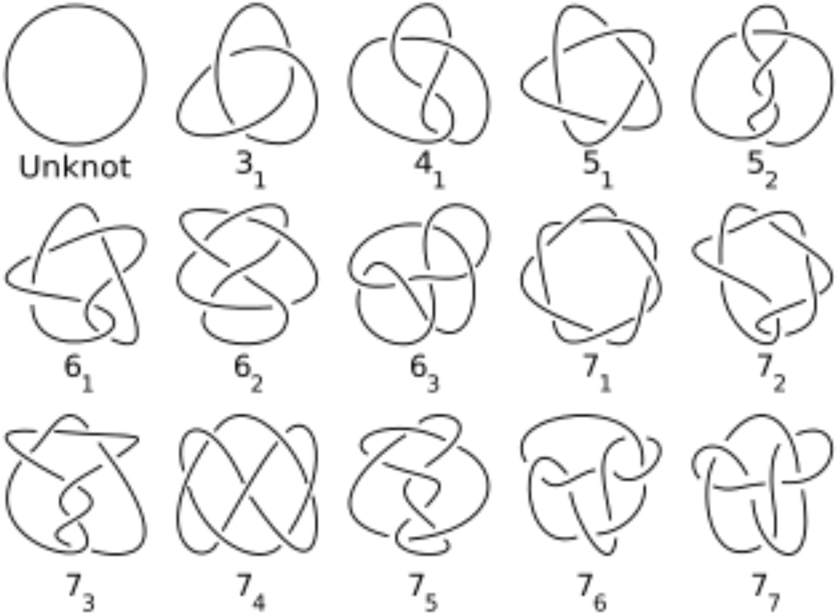
Type 2 move



Type 3 move

Knot Invariants

- A knot invariant is a quantity (in a broad sense) defined for each link from some set which is the same for equivalent links (does not change under Reidemeister moves) from this set.



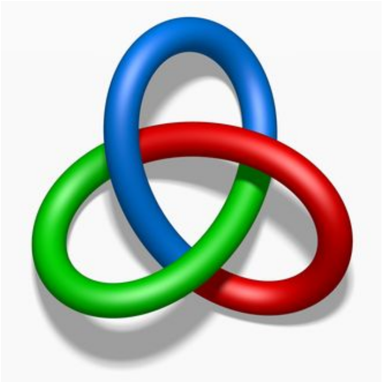
Some knots are organized by the crossing number invariant

Tricolorability

- The **tricolorability** of a knot is the ability of a knot projection to be **colored with three colors** subject to certain rules.

1. At least two colors must be used
2. At each crossing, the three incident strands are either all the same color or all different colors

- As the unknot is not tricolorable, the **trefoil knot is not equal to the unknot**.

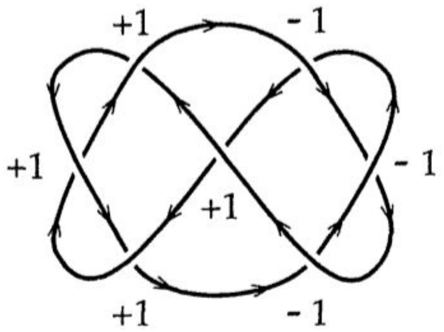


A tricolored trefoil knot

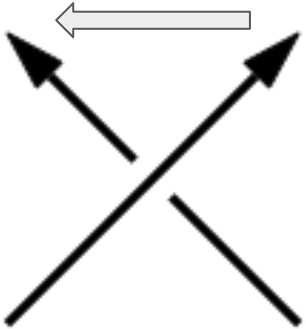
The unknot

Writhe number

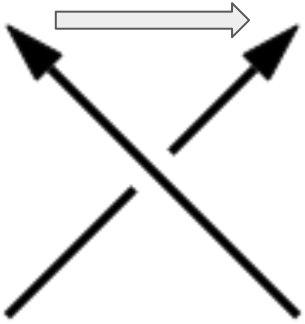
- Define the **writhe** w(L) as the sum of the "crossings" of the link L.
- Only a type 1 Reidemeister move can change w(L) (by +-1).



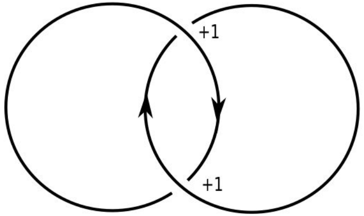
w(L) = +4-3 = 1



+1 crossing



-1 crossing

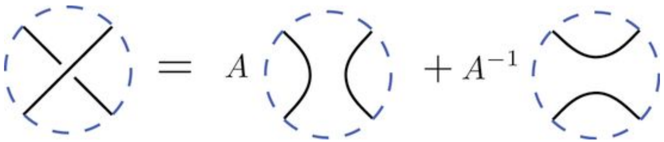


Example of oriented link

Jones polynomial

- Define the **bracket polynomial** <L>, as a polynomial with variable A, characterized by the three rules:

$\langle \bigcirc \rangle = 1$, where $\bigcirc$ is the standard notation of the unknot



$$\langle \bigcirc \cup L \rangle = (-A^2 - A^{-2})\langle L \rangle$$

- Then the **auxiliary polynomial** X(L) is defined as:

$$X(L) = (-A^3)^{-w(L)}\langle L \rangle$$

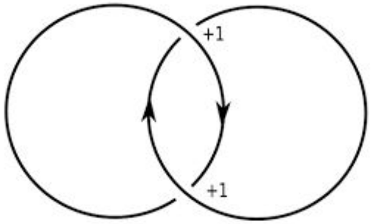
- And the **Jones Polynomial** V(L) is obtained from X by replacing each A with $t^{-1/4}$

Jones polynomial

- An example calculation for the **Hopf link** (replaced with L)

$$\langle\ \rangle = A\langle\ \rangle + A^{-1}\langle\ \rangle$$
$$= A\,(A\langle\ \rangle + A^{-1}\langle\ \rangle) + A^{-1}\,(A\langle\ \rangle + A^{-1}\langle\ \rangle)$$
$$= A\,(A(-(A^2 + A^{-2})) + A^{-1}\,(1)) + A^{-1}\,(A(1) + A^{-1}\,(-(A^2 + A^{-2})))$$
$$= -A^4 - A^{-4}$$

w(L) = 2



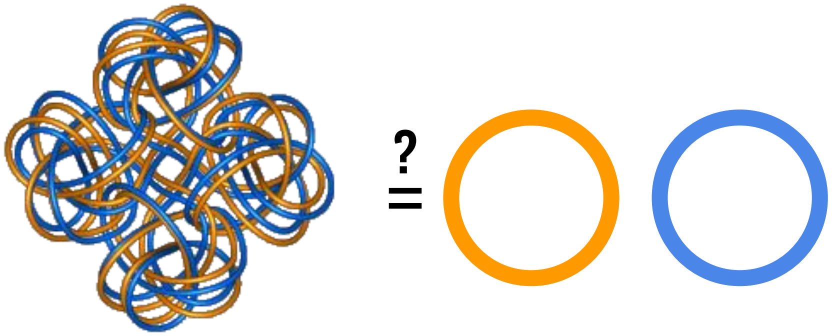
The writhe number of the Hopf link

$$X(L) = (-A^3)^{-2}(-A^4 - A^{-4}) = -A^{-2} - A^{-10}$$

$$V(L) = -t^{1/2} - t^{5/2}$$

A Problem with the Jones Polynomial

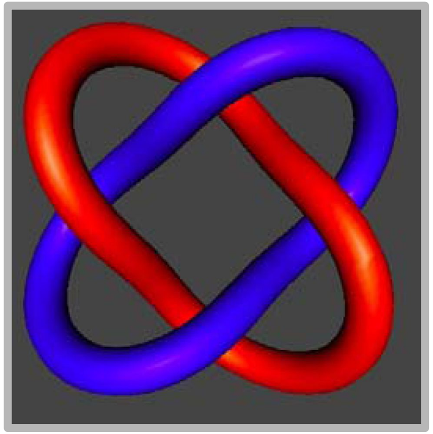
- The Jones Polynomial is **dependent** on the bracket polynomial.
- For every crossing considered, the number of diagrams increase twice. That means the Jones Polynomial becomes **exponentially hard to calculate** (2^n).



Is this link equivalent to an unlink of 2 components?

Linking Number

- The linking number is a knot invariant for a 2-component link that measures the **degree of linkage** between two closed curves or components in three-dimensional space.
- Intuitively, it represents the extent to which the two curves are entangled with each other.
- Typically denoted by ***lk(K,J)***, where K and J are knots.



(2,4)-torus link

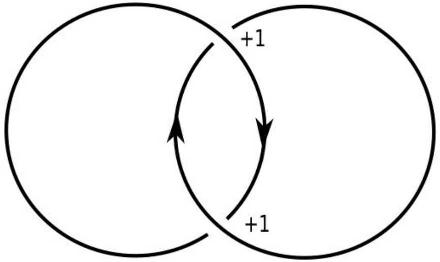
lk( ) Calculation

- Assign both components with an orientation. Each crossing between both components will be **clockwise** or **counterclockwise**. Give each crossing a value:

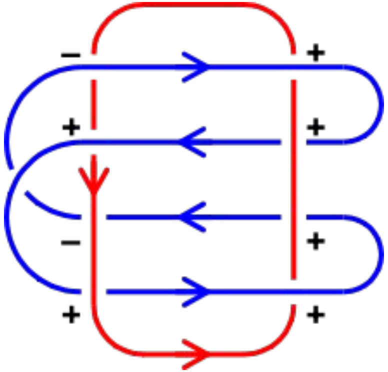
  $$c_i = \pm 1$$

- $$lk(K, J) = \frac{1}{2} \sum_{i=1}^{n} c_i$$

The Hopf link

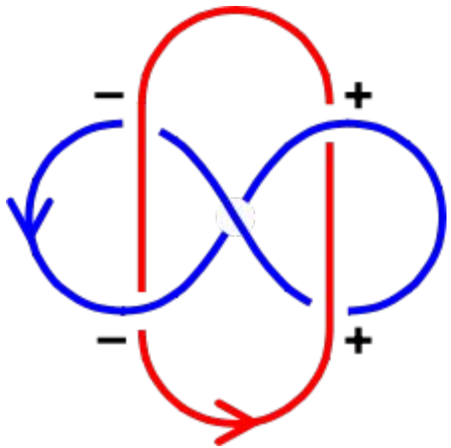


A link with lk() = +4



Limitations of lk( )

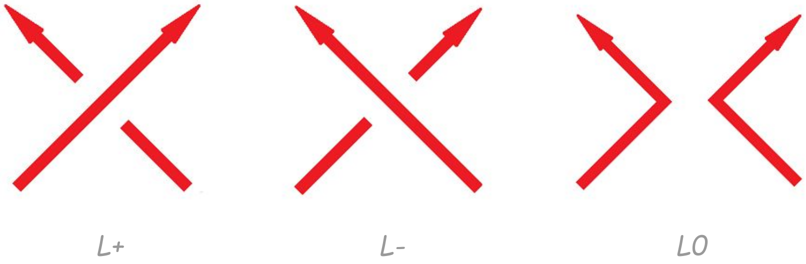
- Any link that can be transformed into an unlink has a linking number of 0 but it's not true vice versa.
- An example of this is the **Whitehead link**.
- The linking number can only be computed for **two components** at a time.



The Whitehead link

The Conway Polynomial

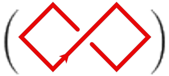
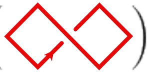
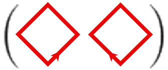
- Is denoted by $\nabla(z)$
- Satisfies the following two skein relations:



$$\nabla(O) = 1$$

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$$

- Let's do a simple calculation.

$\nabla($  $) - \nabla($  $) = z\nabla($  $) \Rightarrow \nabla(U_2) = 0$

- Similarly (n > 1): $\nabla(U_n) = 0$

Properties of its Coefficients (1)

- The Conway Polynomial can be shown in an **expanded form**.

$$\nabla(L) = \sum c_i(L) z^i$$

- After using the skein relation multiple times, we will arrive at a collection of unlinks. The only one which **affects our polynomial is the unknot**.

Theorem

For an n-component link L: $c_i(L) = 0$, for i < n-1.

Properties of its Coefficients (2)

When L is a:

- 1-component link $c_0(L) = 1$
- 2-component link $c_1(L) = lk(L)$
- 3-component link with linking numbers a, b, and c $c_2(L) = ab + bc + ca$

Enhanced Linking Number

Definition

$$\lambda(L) = c_3(L) - c_1(L)(c_2(K) + c_2(J))$$

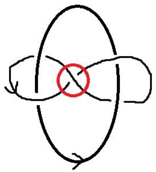
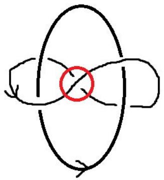
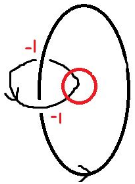
, where K and J are the two knot components of link L.

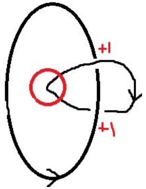
Theorem (Crossing change formula)

$\lambda($  $, J) - \lambda($  $, J) = lk($  $, J)lk($  $, J)$

The Whitehead Link

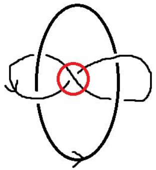
(This one is easy to compute)

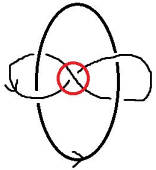
$\lambda\Big($  $\Big) - \lambda\Big($  $\Big) = lk\Big($ 

$\Big) lk\Big($ 

$\Big)$

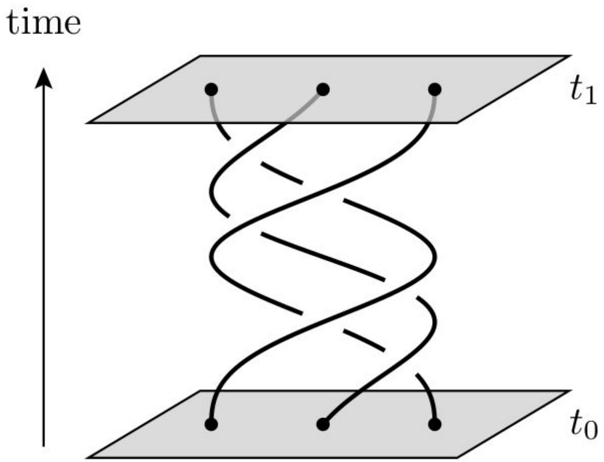
$$\Downarrow$$

$\lambda\Big($  $\Big) - 0 = (-1)(+1)$

$\lambda\Big($  $\Big) = -1$

Another Approach to Solving the Computation Problem

- A **world-line** is the path a particle takes through time (from t_0 to t_1)
- Anyons are particles which exist in 2D. Their properties at the end (t_1) **depend on the knot/link created**.
- Based on this, the **Jones Polynomial can be calculated** for that knot using the anyons.



A knot created by the world-lines of anyons

Acknowledgements

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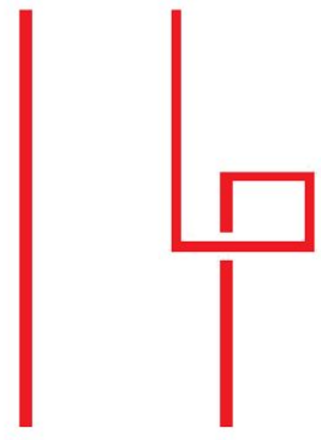
- Dr. Pavel Etingof, Dr. Slava Gerovitch, Dmytro Matvieievskyi and the Yulia's Dream program for providing this valuable opportunity.
- Our families for their constant support.

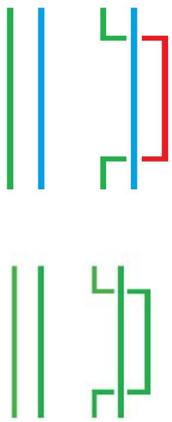
Thank you!

Proof of invariant

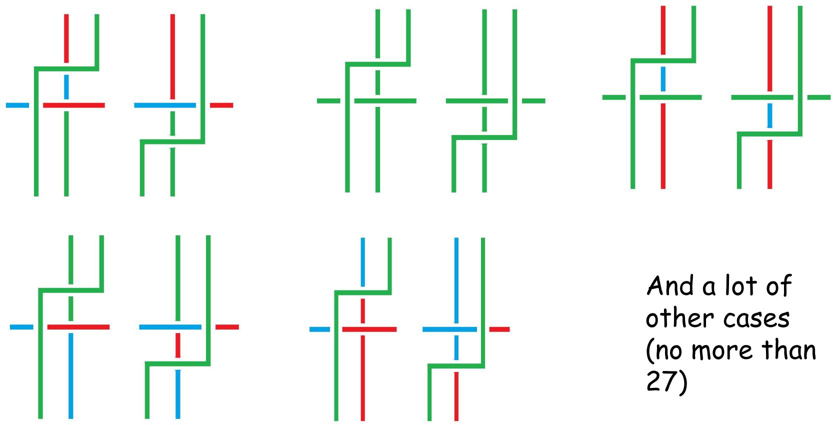
- To prove that tricolorability is an invariant, we must show that it is **not affected by the Reidemeister moves**.



Type 1 move



Type 2 move



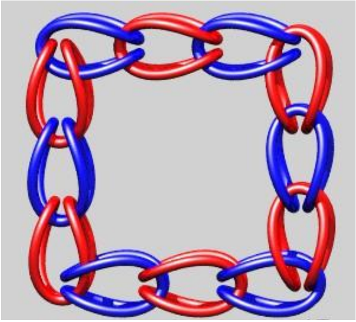
And a lot of other cases (no more than 27)

Type 3 move

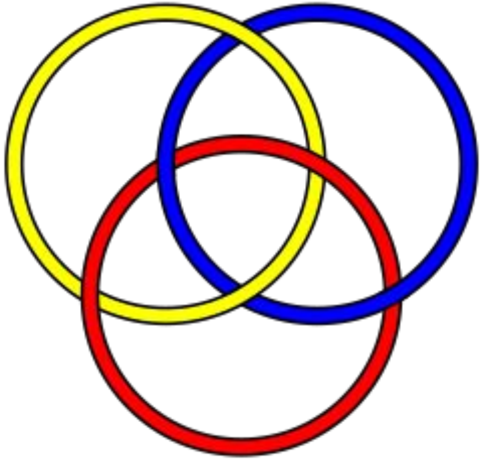
N-component links

- Links of multiple components can be looked at through a **collection of linking numbers**.

- However it will not work if all of the linking numbers are equal to 0. Such a case are the **Brunnian links**.



A 12-component Brunnian link



The Borromean rings