# Elliptic hypergeometric integrals

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## Abstract

Elliptic hypergeometric functions are a new class of special functions that have been developed during these two decades. I will report some recent progresses in the study of elliptic hypergeometric integrals of Selberg type on the basis of collaboration with Masahiko Ito.

## References

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Contents

# 1 $q$-Hypergeometric integrals of Selberg type

## ○ Selberg integral (1942)

Generalization of the beta integral to a multiple integral involving a power of the difference product (Atle Selberg, 1917–2007):

$$\frac{1}{n!}\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} z_i^{\alpha-1}(1-z_i)^{\beta-1} \prod_{1\le i<j\le n} |z_i - z_j|^{2\gamma}\, dz_1 \cdots dz_n$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\alpha+(j-1)\gamma)\,\Gamma(\beta+(j-1)\gamma)\,\Gamma(j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\,\Gamma(\gamma)}$$

Variations and extensions of this formula, including the cases of integrals of trigonometric and elliptic fuctions, provide with foundations for a variety of theories of hypergeometric functions in many variables.

- Hypergeometric integral of Selberg type = Selberg integral in the broad sense
  Integral of powers of polynomials which involves a power of a difference product
  or a Weyl denominator

- Selberg integral in the narrow sense
  Hypergeometric integral of Selberg type which admits an evaluation formula
  in terms of the gamma function

## ○ $q$-Hypergeometric integrals of Selberg type

$z = (z_1, \ldots, z_n)$: coordinates of the $n$-dimensional algebraic torus $\mathbb{T}^n = (\mathbb{C}^*)^n$

There are two types of $q$-hypergeometric integrals (with base $q \in \mathbb{C}^*$, $|q| < 1$):
Jackson integrals /infinite multiple series (Aomoto–Ito),
*versus* ordinary integrals over $n$-cycles in $\mathbb{T}^n$ (Macdonald)

• **Jackson integral:** With a base point $\zeta = (\zeta_1, \ldots, \zeta_n) \in (\mathbb{C}^*)^n$, the Jackson integral of a function $\varphi(z)$ is defined as the infinite multiple series

$$\frac{1}{(1-q)^n} \int_0^{\zeta_1 \infty} \cdots \int_0^{\zeta_n \infty} \varphi(z_1, \ldots, z_n) \, \frac{d_q z_1 \cdots d_q z_n}{z_1 \cdots z_n} = \sum_{\nu_1 = -\infty}^{\infty} \cdots \sum_{\nu_n = -\infty}^{\infty} \varphi(q^{\nu_1} \zeta_1, \ldots, q^{\nu_n} \zeta_n).$$

In the notation of multi-indices $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$, $q^\nu \zeta = (q^{\nu_1} \zeta_1, \ldots, q^{\nu_n} \zeta_n) \in (\mathbb{C}^*)^n$,

$$\int_0^{\zeta \infty} \varphi(z) \, \omega_q(z) = \sum_{\nu \in \mathbb{Z}^n} \varphi(\zeta q^\nu), \quad \omega_q(z) = \frac{1}{(1-q)^n} \frac{d_q z_1 \cdots d_q z_n}{z_1 \cdots z_n}.$$

Sum of the values of $\varphi(z)$ over the multiplicative lattice $\Lambda_\zeta = q^{\mathbb{Z}^n} \zeta \subset (\mathbb{C}^*)^n$.

• **Ordinary integral over an $n$-cycle:**

$$\int_C \varphi(z) \, \omega(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_C \varphi(z_1, \ldots, z_n) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}, \quad \omega(z) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}$$

Typically, the real torus $\mathbb{T}^n_{\mathbb{R}} = \{|z_1| = \cdots = |z_n| = 1\}$ is chosen for the $n$-cycle $C$.

# ○ *q*-Shifted factorials

## • *q*-Shifted factorials:

$$(z;q)_\infty = \prod_{i=0}^{\infty}(1-q^i z), \quad (z;q)_k = \frac{(z;q)_\infty}{(q^k z;q)_\infty} \quad (k \in \mathbb{Z})$$

For $k = 0, 1, 2, \ldots$,

$$(z;q)_k = (1-z)(1-qz)\cdots(1-q^{k-1}z), \quad (z;q)_{-k} = \frac{1}{(1-q^{-k}z)(1-q^{-k+1}z)\cdots(1-q^{-1}z)}.$$

$q$-Shifted factorials are regarded as counterparts of *power functions* or *gamma functions*:

$$\frac{(q^\beta z;q)_\infty}{(q^\alpha z;q)_\infty} \;\to\; (1-z)^{\alpha-\beta}; \qquad \frac{(q;q)_\infty}{(q^s;q)_\infty}(1-q)^{1-s} \;\to\; \Gamma(s)$$

For $k \in \mathbb{Z}$ or $k = \infty$, a product of $q$-shifted factorials are often abbreviated as

$$(a_1, \ldots, a_r; q)_k = (a_1;q)_k \cdots (a_r;q)_k.$$

## ○ $q$-Beta and $q$-hypergeometric integrals (contour integrals)

- **Askey–Wilson $q$-beta integral:** double sign: $f(z^{\pm 1}) = f(z)f(z^{-1})$

$$\frac{1}{2\pi\sqrt{-1}}\int_C \frac{(z^{\pm 2};q)_\infty}{(az^{\pm 1}, bz^{\pm 1}, cz^{\pm 1}, dz^{\pm 1};q)_\infty}\frac{dz}{z} = \frac{2}{(q;q)_\infty}\frac{(abcd;q)_\infty}{(ab,ac,ad,bc,bd,cd;q)_\infty}$$

$C$: a closed curve separating the poles accumulating at $z=0$ and those at $z=\infty$.

- **Nassrallah–Rahman $q$-beta integral:** Under the condition $a_0a_1\cdots a_5 = q$,

$$\frac{1}{2\pi\sqrt{-1}}\int_C \frac{(z^{\pm 2};q)_\infty(qa_0^{-1}z^{\pm 1};q)_\infty}{\prod_{k=1}^{5}(a_kz^{\pm 1};q)_\infty}\frac{dz}{z} = \frac{2}{(q;q)_\infty}\frac{\prod_{i=1}^{5}(q/a_ia_0;q)_\infty}{\prod_{1\le i<j\le 5}(a_ia_j;q)_\infty}$$

- **Rahman's $q$-hypergeometric integral:** (Rahman 1986)

Under the balancing condition $a_0a_1\cdots a_7 = q^2$,

$$\prod_{1\le i<j\le 6}(a_ia_j;q)_\infty\cdot\frac{(q;q)_\infty}{4\pi\sqrt{-1}}\int_C\frac{(z^{\pm 2};q)_\infty\prod_{i=0,7}(qa_i^{-1}z^{\pm 1};q)_\infty}{\prod_{i=1}^{6}(a_iz^{\pm 1};q)_\infty}\frac{dz}{z}$$
$$= \frac{\prod_{i=1}^{6}(qa_i/a_0;q)_\infty(q/a_ia_7;q)_\infty}{(q^2a_0^2;q)_\infty(a_0/a_7;q)_\infty}\,{}_{10}W_9\big(q/a_0^2;q/a_0a_1,q/a_0a_2,\ldots,q/a_0a_7;q,q\big)$$
$$+\frac{\prod_{i=1}^{6}(qa_i/a_7;q)_\infty(q/a_ia_0;q)_\infty}{(q^2a_7^2;q)_\infty(a_7/a_0;q)_\infty}\,{}_{10}W_9\big(q/a_7^2;q/a_0a_7,q/a_1a_7,\ldots,q/a_6a_7;q,q\big).$$

$${}_{r+3}W_{r+2}\big(a_0;a_1,\ldots,a_r;q,z\big) = \sum_{k=0}^{\infty}\frac{1-q^{2k}a_0}{1-a_0}\frac{(a_0;q)_k}{(q;q)_k}\prod_{i=1}^{r}\frac{(a_i;q)_k}{(qa_0/a_i;q)_k}\;z^k$$

## ○ $q$-Hypergeometric integral of Selberg type

$z=(z_1,\dots,z_n)$: coordinates of the algebraic torus $\mathbb{T}^n=(\mathbb{C}^*)^n$

- **Gustafson's $q$-Selberg integral** (1990)

[Askey–Wilson] For generic complex parameters $a=(a_1,\dots,a_4)$ and $t$,

$$\frac{1}{(2\pi\sqrt{-1})^n}\int_{C^n}\prod_{i=1}^{n}\frac{(z_i^{\pm 2};q)_\infty}{\prod_{k=1}^{4}(a_k z_i^{\pm 1};q)_\infty}\prod_{1\le i<j\le n}\frac{(z_i^{\pm 1}z_j^{\pm 1};q)_\infty}{(t z_i^{\pm 1}z_j^{\pm 1};q)_\infty}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

$$=\frac{2^n n!}{(q;q)_\infty^n}\prod_{i=1}^{n}\left(\frac{(t;q)_\infty}{(t^i;q)_\infty}\frac{(a_1a_2a_3a_4t^{n+i-2};q)_\infty}{\prod_{1\le k<l\le 4}(t^{i-1}a_ka_l;q)_\infty}\right)$$

The integrand is the weight function for the *Koornwinder polynomials* ($BC_n$).

[Nassrallah-Rahman] Under the balancing condition $a_0a_1a_2a_3a_4a_5t^{2n-2}=q^2$,

$$\frac{1}{(2\pi\sqrt{-1})^n}\int_{C^n}\prod_{i=1}^{n}\frac{(z_i^{\pm 2};q)_\infty(qa_0^{-1}z_i^{\pm 1};q)_\infty}{\prod_{k=1}^{5}(a_k z_i^{\pm 1};q)_\infty}\prod_{1\le i<j\le n}\frac{(z_i^{\pm 1}z_j^{\pm 1};q)_\infty}{(t z_i^{\pm 1}z_j^{\pm 1};q)_\infty}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

$$=\frac{2^n n!}{(q;q)_\infty^n}\prod_{i=1}^{n}\left(\frac{(t;q)_\infty}{(t^i;q)_\infty}\frac{\prod_{k=1}^{5}(t^{1-i}q/a_0a_k;q)_\infty}{\prod_{1\le k<l\le 5}(t^{i-1}a_ka_l;q)_\infty}\right)$$

# 2 Elliptic hypergeometric integrals of Selberg type

## ○ Ruijsenaars' elliptic gamma function

With two (generic) bases $p, q \in \mathbb{C}^*$, $|p| < 1, |q| < 1$,

$$\Gamma(z;p,q) = \frac{(pq/z;p,q)_\infty}{(z;p,q)_\infty}, \quad (z;p,q)_\infty = \prod_{i,j=0}^{\infty}(1-p^i q^j z).$$

It is a meromorphic function on $\mathbb{C}^*$ with simple poles at $z = p^{-i}q^{-j}$ $(i, j = 0, 1, \ldots)$.

- Jacobi theta function (in the multiplicative variable):

$$\theta(z;p) = (z;p)_\infty (p/z;p)_\infty; \quad \theta(pz;p) = -z^{-1}\theta(z;p), \quad \theta(p/z;p) = \theta(z;p)$$

- The elliptic gamma function satisfies the following functional equations:

$$\Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q), \quad \Gamma(pq/z;p,q) = \Gamma(z;p,q)^{-1}$$

- In the double sign notation $f(z^{\pm 1}) = f(z)f(z^{-1})$,

$$\begin{aligned}\frac{1}{\Gamma(z^{\pm 1};p,q)} &= \frac{(z^{\pm 1};p,q)_\infty}{(pqz^{\pm 1};p,q)_\infty} = (1-z^{\pm 1})(pz^{\pm 1};p)_\infty (qz^{\pm 1};q)_\infty \\ &= -z^{-1}(z, p/z;p)_\infty (z, q/z;q)_\infty = -z^{-1}\theta(z;p)\theta(z;q)\end{aligned}$$

  holomorphic on $\mathbb{C}^*$, splits into the product of two theta functions with bases $p$, $q$.

- In the limit as $p \to 0$,

$$\theta(z;p) \to (1-z), \quad \Gamma(z;p,q) \to \frac{1}{(z;q)_\infty}, \quad \Gamma(pz;p,q) \to (q/z;q)_\infty$$

## ○ Elliptic hypergeometric integral of Selberg type ($BC_n$)

**• Elliptic beta integral** (Spiridonov 2001)

Under the balancing condition $a_1 \cdots a_6 = pq$,

$$\frac{(p;p)_\infty (q;q)_\infty}{4\pi\sqrt{-1}} \int_C \frac{\prod_{k=1}^{6} \Gamma(a_k z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z} = \prod_{1 \le k < l \le 6} \Gamma(a_k a_l; p, q)$$

- Elliptic extension of the Nassrallah–Rahman $q$-beta integral
- Integral version of the Frenkel–Turaev sum

**• Elliptic hypergeometric integral of Selberg type**

The following integral is called the $BC_n$ *elliptic hypergeometric integral of Selberg type*:

$$I_n(a) = \int_{C^n} \Phi(z;a)\omega(z), \quad \omega(z) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}$$

$$\Phi(z;a) = \prod_{i=1}^{n} \frac{\prod_{k=1}^{m} \Gamma(a_k z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \le i < j \le n} \frac{\Gamma(t z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)}$$

$$a = (a_1, \ldots, a_m) \in (\mathbb{C}^*)^m, \quad t \in \mathbb{C}^*$$

· When $|a_k| < 1$ $(k = 1, \ldots, m)$, $|t| < 1$, a standard choice for the $n$-cycle $C^n$ is the real torus $\mathbb{T}^n_{\mathbb{R}} = \big\{ |z_1| = \cdots = |z_n| = 1 \big\}$. When the parameters go out from this domain, the $n$-cycle should be deformed accordingly.

• **Elliptic Selberg integral ($m=6$)** (van Diejen-Spiridonov 2001, Rains)
Under the balancing condition $a_1\cdots a_6 t^{2n-2}=pq$,

$$I_n(a_1,\ldots,a_6)=\frac{1}{(2\pi\sqrt{-1})^n}\int_{C^n}\prod_{i=1}^{n}\frac{\prod_{k=1}^{6}\Gamma(a_k z_i^{\pm1};p,q)}{\Gamma(z_i^{\pm2};p,q)}\prod_{1\le i<j\le n}\frac{\Gamma(tz_i^{\pm1}z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

$$=\frac{2^n n!}{(p;p)_\infty^n(q;q)_\infty^n}\prod_{i=1}^{n}\left(\frac{\Gamma(t^i;p,q)}{\Gamma(t;p,q)}\prod_{1\le k<l\le 6}\Gamma(t^{i-1}a_k a_l;p,q)\right)$$

(Elliptic extension of Gustafson's $q$-Selberg integral)

• **$BC_n$ elliptic hypergeometric integral ($m=8$)** (Rains)

$$I_n(a_1,\ldots,a_8)=\frac{1}{(2\pi\sqrt{-1})^n}\int_{C^n}\prod_{i=1}^{n}\frac{\prod_{k=1}^{8}\Gamma(a_k z_i^{\pm1};p,q)}{\Gamma(z_i^{\pm2};p,q)}\prod_{1\le i<j\le n}\frac{\Gamma(tz_i^{\pm1}z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)}\frac{dz_1\cdots dz_n}{z_1\cdots z_n}$$

· The Ruijsenaars–van Diejen difference operator of type $BC_n$ is formally selfadjoint with respect to the scalar product defined by the weight function $\Phi(z)$.

· When $t=q$, the sequence of integrals $I_n(a_1,\ldots,a_8)$ $(n=0,1,2,\ldots)$ provides with a *hypergeometric $\tau$-function* of the $E_8$ elliptic difference Painlevé equation (Rains 2005, Noumi 2018). In this case, $I_n(a_1,\ldots,a_8)$ can also be expressed as an $n\times n$ Casorati deteminant whose entries are elliptic hypergeometric integrals in one variable.

# 3 Determinant of elliptic hypergeometric integrals

## ○ General setting of type $BC_n$

We consider the meromorphic function

$$\Phi(z;a) = \prod_{i=1}^{n} \frac{\prod_{k=1}^{m} \Gamma(a_k z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \prod_{1\le i<j\le n} \frac{\Gamma(t z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)}$$

of $n$ variables $z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n$ with generic parameters $a = (a_1, \ldots, a_m)$ and $t$. The $BC_n$ elliptic hypergeometric integral (of type II) is defined by

$$I_n(a) = \int_{C^n} \Phi(z;a)\omega(z), \quad \omega(z) = \frac{1}{(2\pi\sqrt{-1})^n} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}.$$

The integrand $\Phi(z;a)$ is invariant under the action of the Weyl group $W_n = \{\pm 1\}^n \rtimes \mathfrak{S}_n$ of type $BC_n$ (hyperoctahedral group of degree $n$).

## ○ Bilinear form defined by the integral

Assuming that $m = 2r+4$ (even), we denote by $\mathcal{H}^{(p)}_{r-1}$ the $\mathbb{C}$-vector space of $W_n$-invariant holomorphic functions of degree $r-1$ with respect to $p$:

$$\mathcal{H}^{(p)}_{r-1,n} = \left\{ f \in \mathcal{O}((\mathbb{C}^*)^n)^{W_n} \;\middle|\; T_{p,z_i} f(z) = f(z)(pz_i^2)^{-r+1} \quad (i=1,\ldots,n) \right\}.$$
$$\dim_{\mathbb{C}} \mathcal{H}^{(p)}_{r-1,n} = \tbinom{n+r-1}{r-1}.$$

Taking the two $\mathbb{C}$-vector spaces $\mathcal{H}^{(p)}_{r-1,n}$, $\mathcal{H}^{(q)}_{r-1,n}$ for the two bases $p$, $q$, respectively, we introduce the *hypergeometric pairing* (following the terminology of Tarasov-Varchenko)

$$\langle \cdot, \cdot \rangle_\Phi : \; \mathcal{H}^{(p)}_{r-1,n} \times \mathcal{H}^{(q)}_{r-1,n} \to \mathbb{C},$$
$$\langle \varphi(z), \psi(z) \rangle_\Phi = \int_{C^n} \varphi(z)\psi(z)\Phi(z)\omega(z) \quad (\varphi \in \mathcal{H}^{(p)}_{r-1,n},\; \psi \in \mathcal{H}^{(q)}_{r-1,n})$$

associated with the integral with respect to $\Phi(z) = \Phi(z;a)$.

In this setting the vector space $\mathcal{H}^{(p)}_{r-1,n}$ can be regarded as the space of *$n$-cocycles* representing the $W_n$-invariant $q$-difference de Rham cohomology associated with $\Phi(z)$. The vector space $\mathcal{H}^{(q)}_{r-1,n}$ in turn plays the role of the space of *$n$-cycles* for this $q$-difference de Rham cohomology. Note that the dimension

$$\dim_{\mathbb{C}} \mathcal{H}^{(p)}_{r-1,n} = \dim_{\mathbb{C}} \mathcal{H}^{(q)}_{r-1,n} = \tbinom{n+r-1}{r-1}$$

is 1 for $r=1$, and $n+1$ for $r=2$.

Note that the dimension $\binom{n+r-1}{r-1}$ of $\mathcal{H}^{(p)}_{r-1,n}$ coincides with the cardinality of the set of multiindices

$$Z_{r,n} = \left\{ \mu = (\mu_1, \ldots, \mu_r) \in \mathbb{N}^r \;\middle|\; |\mu| = \mu_1 + \cdots + \mu_r = n \right\}.$$

Choosing generic $r$ parameters $x = (x_1, \ldots, x_r) \in (\mathbb{C}^*)^r$, we consider the set of reference points $(x)_{t,\nu}$ ($\nu \in Z_{r,n}$) in $(\mathbb{C}^*)^n$ defined by *multiple principal specialization*:

$$(x)_{t,\nu} = (x_1, tx_1, \ldots, t^{\nu_1-1}x_1;\ x_2, tx_2, \ldots, t^{\nu_2-1}x_2;\ \ldots) \in (\mathbb{C}^*)^n \quad (r \text{ blocks}).$$

Then one can show that $\mathcal{H}^{(p)}_{r-1,n}$ has a unique *interpolation function basis* such that

$$E_\mu(x; (x)_{t,\nu}; p) = \delta_{\mu,\nu} \quad (\mu, \nu \in Z_{r,n}).$$

Using the two kinds of interpolation functions with bases $p$, $q$ respectively, we define the integrals

$$\begin{aligned} K_{\mu,\nu}(a; x, y) = K_{\mu,\nu}(a; x, y; p, q) &= \langle E_\mu(x; z; p), E_\nu(y; z; q) \rangle_\Phi \\ &= \int_{C^n} E_\mu(x; z; p) E_\nu(y; z; q) \Phi(a; z; p, q) \omega(z) \quad (\mu, \nu \in Z_{r,n}). \end{aligned}$$

The $\binom{n+r-1}{r-1} \times \binom{n+r-1}{r-1}$ matrix $K^{(r,n)}(a; x, y) = \big(K_{\mu,\nu}(a; x, y)\big)_{\mu,\nu \in Z_{r,n}}$ is the representation matrix of the hypergeometric pairing

$$\langle \cdot, \cdot \rangle_\Phi : \mathcal{H}^{(p)}_{r-1,n} \times \mathcal{H}^{(q)}_{r-1,n} \to \mathbb{C}; \quad \langle \varphi(z), \psi(z) \rangle_\Phi = \int_{C^n} \varphi(z) \psi(z) \Phi(z) \omega(z)$$

in terms of the interpolation bases.

We assume below that the balancing condition $a_1 \cdots a_m t^{2n-2} = pq$ is satisfied.

**Theorem A:** *The matrix $K^{(r,n)}(a;x,y)$ satisfies a system of first order $q$-difference and $p$-difference equations of the form*

$$T_{q,a_k} T_{q,a_l}^{-1} K^{(r,n)}(a;x,y) = A_{k,l}(a;x,y)\, K^{(r,n)}(a;x,y) \quad (1 \le k < l \le m),$$
$$T_{p,a_k} T_{p,a_l}^{-1} K^{(r,n)}(a;x,y) = K^{(r,n)}(a;x,y)\, B_{k,l}(a;x,y) \quad (1 \le k < l \le m).$$

We remark that $B_{k,l}(a;x,y)$ is obtained as the transposed matrix of $A_{k,l}(a;x,y)$ with the roles of $(x,y)$ and $(p,q)$ exchanged.

The matrix $K^{(r,n)}(a;x,y)$ can be thought of as a fundamental system of solutions of the $q$-difference/$p$-difference systems. Also, non-degeneracy of the hypergeometric pairing is guaranteed by an explicit evaluation formula for the determinant of $K^{(r,n)}(a;x,y)$.

**Theorem B:** *The determinant of the matrix $K^{(r,n)}(a;x,y)$ is evaluated as follows:*

$$\det K^{(r,n)}(a;x,y) = c^{(r,n)} L^{(r,n)}(a;x,y)$$

$$L^{(r,n)}(a;x,y) = \frac{\prod_{i=0}^{n-1} \prod_{1\le k<l\le m} \Gamma(t^i a_k a_l;p,q)^{\binom{n-i+r-2}{r-1}}}{\prod_{0\le i+j<n} \prod_{1\le k<l\le r} \left(e(t^i x_k, t^j x_l;p) e(t^i y_k, t^j y_l;q)\right)^{\binom{n-i-j+r-3}{r-2}}}$$

$$c^{(r,n)} = \left(\frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n}\right)^{\binom{n+r-1}{r-1}} \frac{\prod_{i=1}^{n} \Gamma(t^i;p,q)^{r\binom{n-i+r-1}{r-1}}}{\Gamma(t;p,q)^{r\binom{n+r-1}{r}}},$$

*where $e(u,v;p) = u^{-1}\theta(uv;p)\theta(u/v;p)$.*

- **$r = 1$ ($m = 6$):** $1 \times 1$ determinant (van Diejen–Spiridonov 2001)

$$\det K^{(1,n)}(a) = \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \frac{\prod_{i=1}^n \Gamma(t^i;p,q)}{\Gamma(t;p,q)^n} \prod_{i=0}^{n-1} \prod_{1\le k<l\le 6} \Gamma(t^i a_k a_l;p,q)$$

- **$r = 2$ ($m = 8$):** $(n+1) \times (n+1)$ determinant

$$\det K^{(2,n)}(a;x,y) = \left(\frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n}\right)^{n+1} \frac{\prod_{i=1}^n \Gamma(t^i;p,q)^{2(n-i+1)}}{\Gamma(t;p,q)^{n(n+1)}} \cdot \frac{\prod_{i=0}^{n-1} \prod_{1\le k<l\le 8} \Gamma(t^i a_k a_l;p,q)^{n-i}}{\prod_{0\le i+j<n} e(t^i x_1, t^j x_2;p) e(t^i y_1, t^j y_2;q)}.$$

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