Horn's problem and projection of orbital measures for unitary and pseudounitary groups

Jacques Faraut

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1. Horn's problem, and Horn's conjecture

A and B are $n \times n$ Hermitian matrices, and $C = A + B$.
Assume that the eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ of A,
and the eigenvalues $\beta_1 \geq \cdots \geq \beta_n$ of B are known.
Horn's problem : What can be said about the eigenvalues $\gamma_1 \geq \cdots \geq \gamma_n$ of $C = A + B$?

Weyl's inequalities (1912)

$$\begin{array}{lcll} \gamma_{i+j-1} & \leq & \alpha_i + \beta_j & \text{for } i + j \leq n + 1, \\ \gamma_{i+j-n} & \geq & \alpha_i + \beta_j & \text{for } i + j \geq n + 1. \end{array}$$

Lemma $\mathcal{U}, \mathcal{V}, \mathcal{W}$ subspaces of $\mathbb{R}^n$. If

$$\dim \mathcal{U} + \dim \mathcal{V} + \dim \mathcal{W} \geq 2n+1,$$

then $\mathcal{U} \cap \mathcal{V} \cap \mathcal{W} \neq \{0\}$.

Proof of Weyl's inequalities

Eigenvectors $u_1, \ldots, u_n$ of A, $v_1, \ldots, v_n$ of B, $w_1, \ldots, w_n$ of C.
For $i + j \leq n+1$, define

$$\begin{array}{ll} \mathcal{U} = Vect(u_i, \ldots, u_n), & \dim \mathcal{U} = n - i + 1, \\ \mathcal{V} = Vect(v_j, \ldots, v_n), & \dim \mathcal{V} = n - j + 1, \\ \mathcal{W} = Vect(w_1, \ldots, w_{i+j-1}), & \dim \mathcal{W} = i + j - 1. \end{array}$$

Then $\dim \mathcal{U} + \dim \mathcal{V} + \dim \mathcal{W} = 2n+1$. By the lemma $\mathcal{U} \cap \mathcal{V} \cap \mathcal{W} \neq \{0\}$. Take $x \in \mathcal{U} \cap \mathcal{V} \cap \mathcal{W}$, $\|x\| = 1$. Then

$$(Ax \mid x) \leq \alpha_i, \ (Bx \mid x) \leq \beta_j, \ (Cx \mid x) \geq \gamma_{i+j-1}.$$

Hence

$$\gamma_{i+j-1} \leq (Cx \mid x) = (Ax \mid x) + (Bx \mid x) \leq \alpha_i + \beta_j.$$

Horn's conjecture (1962)

The set of possible eigenvalues $\gamma_1 \geq \cdots \geq \gamma_n$ for $C = A + B$ is determined by a family of inequalities of the form

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j,$$

for certain admissible triples (I, J, K) of subsets of $\{1, \ldots, n\}$.
Klyachko has proven Horn's conjecture,
and described these admissible triples **(1998)**.
We will call this set Horn's set $\mathrm{Horn}(\alpha, \beta)$.

$n = 3$, $\alpha = (3.5, 1.4, -4.9)$, $\beta = (2, -0.86, -1.14)$.

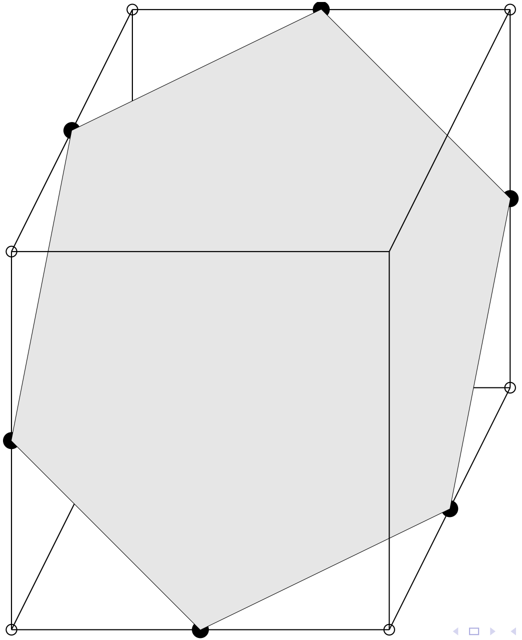
Weyl'inequalities gives

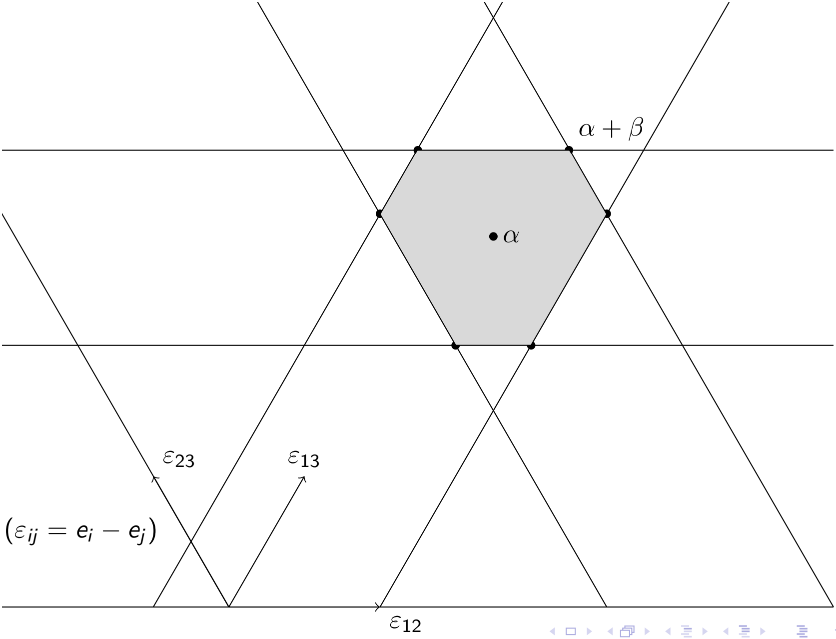
$$\begin{aligned} a_1 &\leq \gamma_1 \leq b_1 \\ a_2 &\leq \gamma_2 \leq b_2 \\ a_3 &\leq \gamma_3 \leq b_3 \end{aligned}$$

In the plane

$$x_1 + x_2 + x_3 = 0,$$

these inequalities determine a hexagon.




$\alpha + \beta$
α
ε_{23}
ε_{13}
$(\varepsilon_{ij} = e_i - e_j)$
ε_{12}

We consider Horn's problem from a probabilistic viewpoint.
The set of Hermitian matrices X with spectrum $\{\alpha_1, \dots, \alpha_n\}$ is an orbit $\mathcal{O}_\alpha$ for the natural action of the unitary group $U(n)$:
$X \mapsto UXU^*$ $(U \in U(n))$.
Assume that
the random Hermitian matrix X is uniformly distributed on the orbit $\mathcal{O}_\alpha$, and the random Hermitian matrix Y uniformly distributed on $\mathcal{O}_\beta$.

What is the joint distribution of the eigenvalues of the sum $Z = X + Y$?

This distribution is a probability measure on $\mathbb{R}^n$ that we will determine explicitly.

Orbits for the action of $U(n)$ on $\mathcal{H}_n(\mathbb{C})$

Spectral theorem : The eigenvalues of a matrix $A \in \mathcal{H}_n(\mathbb{C})$ are real and the eigenspaces are orthogonal.
The unitary group $U(n)$ acts on $\mathcal{H}_n(\mathbb{C})$ by the transformations

$$X \mapsto UXU^*$$

For $A = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$, consider the orbit

$$\mathcal{O}_\alpha = \{UAU^* \mid U \in U(n)\}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n.$$

By the spectral theorem

$$\mathcal{O}_\alpha = \big\{X \in \mathcal{H}_n(\mathbb{C}) \mid \mathrm{spectrum}(X) = \{\alpha_1, \dots, \alpha_n\}\big\}$$

Orbital measures, and radial part of an invariant measure

The orbit $\mathcal{O}_\alpha$ carries a natural probability measure:
the *orbital measure* μ_α,
image of the normalized Haar measure ω of the compact group $U(n)$ by the map $U \mapsto UAU^*$. For a function f on $\mathcal{O}_\alpha$,

$$\int_{\mathcal{O}_\alpha} f(X)\mu_\alpha(dX) = \int_{U(n)} f(UAU^*)\omega(dU).$$

A $U(n)$-invariant measure μ on $\mathcal{H}_n(\mathbb{C})$ can be seen as an integral of orbital measures:
it can be written

$$\int_{\mathbb{H}_n(\mathbb{C})} f(X)\mu(dX) = \int_{\mathbb{R}^n} \Big(\int_{U(n)} f(U\mathrm{diag}(t_1, \ldots, dt_n)U^*)\omega(dU) \Big) \nu(dt),$$

where ν is a $\mathfrak{S}_n$-invariant measure on $\mathbb{R}^n$, called the *radial part* of μ.

If μ is a $U(n)$-invariant probability measure on $\mathcal{H}_n(\mathbb{C})$,
and X a random Hermitian matrix with law μ,
then the joint distribution of the eigenvalues of X
is the radial part ν of μ.

Assume that the random Hermitian matrix X is uniformly distributed
on the orbit $\mathcal{O}_\alpha$, i.e. with law μ_α,
and Y uniformly distributed on $\mathcal{O}_\beta$, i.e. with law μ_β,
then the law of the sum $Z = X + Y$ is the convolution product $\mu_\alpha * \mu_\beta$,
and the joint distribution of the eigenvalues of Z
is the radial part $\nu_{\alpha,\beta}$ of the measure $\mu = \mu_\alpha * \mu_\beta$.

Hence the problem is to determine this radial part $\nu_{\alpha,\beta}$. Then Horn's set is given by

$$\mathrm{Horn}(\alpha, \beta) = \mathrm{support}(\nu_{\alpha,\beta}) \cap C_n,$$

where

$$C_n = \{x \in \mathbb{R}^n \mid x_1 \geq \cdots \geq x_n\}.$$

Fourier-Laplace transform

For a bounded measure μ on $\mathcal{H}_n(\mathbb{C})$,

$$\mathcal{F}\mu(Z) = \int_{\mathcal{H}_n(\mathbb{C})} e^{\mathrm{tr}\,(ZX)} \mu(dX).$$

If μ is $U(n)$-invariant, then $\mathcal{F}\mu$ is $U(n)$-invariant as well, and hence determined by its restriction to the subspace D_n of diagonal matrices.

For $Z = \mathrm{diag}(z_1, \ldots, z_n)$, $T = \mathrm{diag}(t_1, \ldots, t_n)$, define

$$\mathcal{E}(z,t) := \int_{U(n)} e^{\mathrm{tr}\,(ZUTU^*)} \omega(dU).$$

Then $\mathcal{F}\mu_\alpha(Z) = \mathcal{E}(z, \alpha)$.

If μ is $U(n)$-invariant,

$$\mathcal{F}\mu(Z) = \int_{\mathbb{R}^n} \mathcal{E}(z,t)\nu(dt),$$

where ν is the radial part of μ.

Taking $\mu = \mu_\alpha * \mu_\beta$,

$$\mathcal{E}(z,\alpha)\mathcal{E}(z,\beta) = \int_{\mathbb{R}^n} \mathcal{E}(z,t)\nu_{\alpha,\beta}(dt).$$

Harish-Chandra-Itzykson-Zuber formula

A is a Hermitian matrix with eigenvalues $\alpha_1, \ldots, \alpha_n$,
and B with eigenvalues $\beta_1, \ldots, \beta_n$.

$$\int_{U(n)} e^{\mathrm{tr}\,(AUBU^*)} \omega(dU) = \delta_n! \frac{1}{V_n(\alpha) V_n(\beta)} \det\left(e^{\alpha_i \beta_j}\right)_{1 \leq i,j \leq n}$$

V_n is the Vandermonde polynomial: for $x = (x_1, \ldots, x_n)$,

$$V_n(x) = \prod_{i<j} (x_i - x_j)$$

and

$$\delta_n = (n-1, n-2, \ldots, 2, 1, 0), \quad \delta_n! = (n-1)!(n-2)! \ldots 2!$$

Heckman's measure

Consider the projection $\mathrm{Proj} : \mathcal{H}_n(\mathbb{C}) \to D_n$ onto the subspace D_n of real diagonal matrices.

Horn's theorem The projection $\mathrm{Proj}(\mathcal{O}_\alpha)$ of the orbit $\mathcal{O}_\alpha$ is the convex hull of the points $\sigma(\alpha)$

$$\mathrm{Proj}(\mathcal{O}_\alpha) := \mathrm{Conv}(\{\sigma(\alpha) \mid \sigma \in \mathfrak{S}_n\})$$

$(\sigma(\alpha) = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}))$

Heckman's measure is the projection $M_\alpha = \mathrm{Proj}(\mu_\alpha)$ of the orbital measure μ_α.

It is a probability measure on $\mathbb{R}^n$, symmetric, i.e. $\mathfrak{S}_n$-invariant, with compact support: $\mathrm{support}(M_\alpha) = \mathrm{Conv}(\mathfrak{S}_n\alpha)$.

Fourier-Laplace transform of a bounded measure M on $\mathbb{R}^n$:

$$\widehat{M}(z) = \int_{\mathbb{R}^n} e^{(z|x)} M(dx)$$

The Fourier-Laplace transform of Heckman's measure M_α is the restriction to D_n of the Fourier-Laplace transform of the orbital measure μ_α:
for $Z = \mathrm{diag}(z_1, \ldots, z_n)$,

$$\widehat{M_\alpha}(z) = \mathcal{F}\mu_\alpha(Z)$$

Therefore $\widehat{M_\alpha}(z) = \mathcal{E}(z, \alpha)$,
and by the Harish-Chandra-Itzykson-Zuber formula,

$$\widehat{M_\alpha}(z) = \delta_n! \frac{1}{V_n(z) V_n(\alpha)} \det\big(e^{z_i \alpha_j}\big)_{1 \le i,j \le n}$$

Define the skew-symmetric measure

$$\eta_\alpha = \frac{\delta_n!}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\delta_{\sigma(\alpha)}$$

($\varepsilon(\sigma)$ is the signature of the permutation σ).
Fourier-Laplace of η_α:

$$\widehat{\eta_\alpha}(z) = \frac{\delta_n!}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)e^{(z|\sigma(\alpha))} = \frac{\delta_n!}{V_n(\alpha)} \det\big(e^{z_i\alpha_j}\big)_{1\leq i,j\leq n}$$

By the Harish-Chandra-Itzykson-Zuber formula

$$\widehat{\eta_\alpha}(z) = V_n(z)\widehat{M_\alpha}(z).$$

Proposition

$$V_n\Big(-\frac{\partial}{\partial x}\Big)M_\alpha = \eta_\alpha$$

Elementary solution of $V_n\left(\frac{\partial}{\partial x}\right)$

Proposition Define the distribution E_n on $\mathbb{R}^n$

$$\langle E_n, \varphi\rangle = \int_{\mathbb{R}_+^{\frac{n(n-1)}{2}}} \varphi\Big(\sum_{i<j} t_{ij}\varepsilon_{ij}\Big) dt_{ij}$$

$(\varepsilon_{ij} = e_i - e_j)$ Then

$$V_n\left(\frac{\partial}{\partial x}\right)E_n = \delta_0.$$

Proof: An elementary solution of the first order differential operator $\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}$ is the Heaviside distribution

$$\langle Y_{ij}, \varphi\rangle = \int_0^\infty \varphi(t\varepsilon_{ij})dt$$

Hence

$$E_n = \prod_{i<j}^{*} Y_{ij}$$

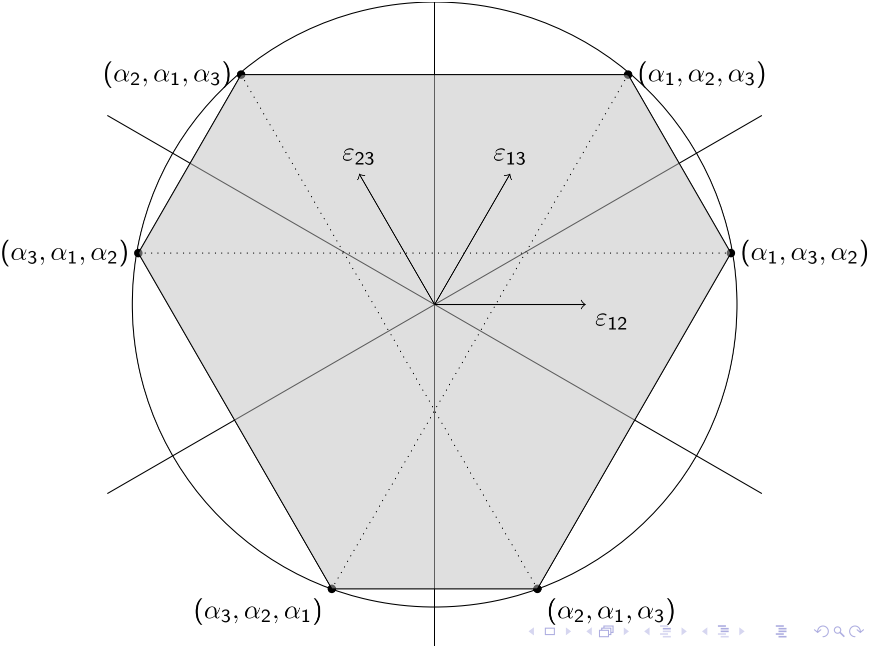
is an elementary solution of $V_n\left(\frac{\partial}{\partial x}\right)$.

Theorem

$$M_\alpha = \check{E}_n * \eta_\alpha$$

$(\check{\varphi}(x) = \varphi(-x)$, $\langle \check{E}_n, \varphi \rangle = \langle E_n, \check{\varphi} \rangle)$
Heckman's measure M_α is supported by the hyperplane
$x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n$.

Next figure is for $n = 3$,
drawn in the plane $x_1 + x_2 + x_3 = \alpha_1 + \alpha_2 + \alpha_3$.



$(\alpha_2, \alpha_1, \alpha_3)$
$(\alpha_1, \alpha_2, \alpha_3)$
ε_{23}
ε_{13}
$(\alpha_3, \alpha_1, \alpha_2)$
$(\alpha_1, \alpha_3, \alpha_2)$
ε_{12}
$(\alpha_3, \alpha_2, \alpha_1)$
$(\alpha_2, \alpha_1, \alpha_3)$

The radial part $\nu_{\alpha,\beta}$

Recall
X is a random Hermitian matrix on $\mathcal{O}_\alpha$ with law μ_α,
and Y on $\mathcal{O}_\beta$ with law μ_β.
The joint distribution of the eigenvalues of $Z = X + Y$ is the radial part $\nu_{\alpha,\beta}$ of $\mu_\alpha * \mu_\beta$.
Theorem(J.F.,2019)

$$\begin{aligned} \nu_{\alpha,\beta} &= \frac{1}{n!}\frac{1}{\delta_n!} V_n(x)\ \eta_\alpha * M_\beta \\ &= \frac{1}{n!}\frac{1}{\delta_n!}\frac{V_n(x)}{V_n(\alpha)} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\delta_{\sigma(\alpha)} * M_\beta. \end{aligned}$$

The sum has positive and negative terms.
However $\nu_{\alpha;\beta}$ is a probability measure on $\mathbb{R}^n$.
The measure $\nu_{\alpha,\beta}$ is symmetric (invariant by $\mathfrak{S}_n$).

The set of possible systems of eigenvalues for the sum $Z = X + Y$ is

$$\mathrm{support}(\nu_{\alpha,\beta})$$

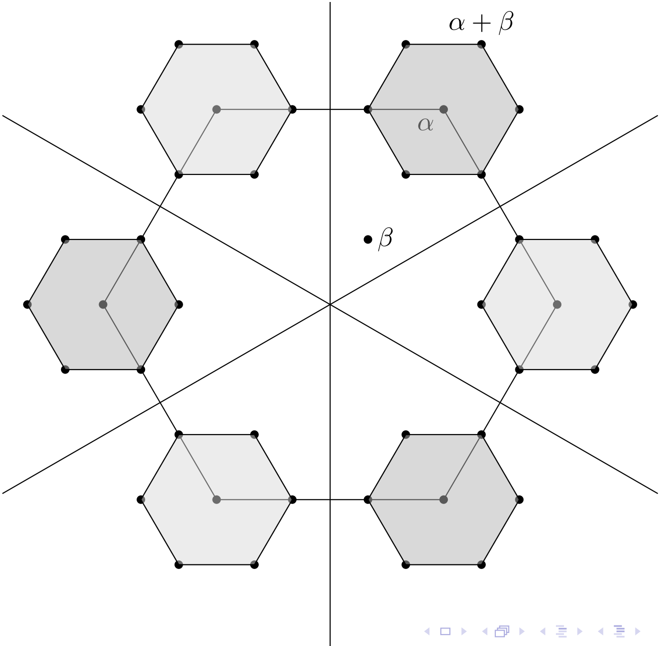
The proof amounts to check that the measure

$$\nu = \frac{1}{n!}\frac{1}{\delta_n!} V_n(x)\eta_\alpha * M_\beta$$

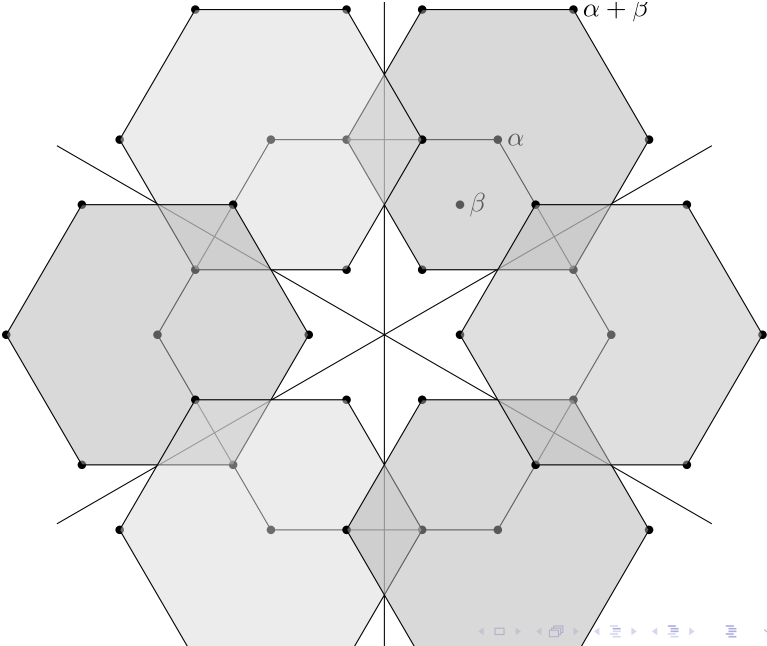
satisfies the relation

$$\int_{\mathbb{R}^n} \mathcal{E}(z,t)\nu(dt) = \mathcal{E}(z,\alpha)\mathcal{E}(z,\beta)$$

Next figure is for $n = 3$, $\alpha = (3, 0, -3)$, $\beta = (1, 0, -1)$


$\alpha + \beta$
α
β

Next figure is for $n = 3$, $\alpha = (3, 0, -3)$, $\beta = (2, 0, -2)$


$\alpha + \beta$
α
β

In the first case the condition

$$\sup|\beta_i - \beta_j| < \inf_{i \neq j} |\alpha_i - \alpha_j|$$

is satisfied, and, under that condition,

$$\mathrm{Horn}(\alpha, \beta) = \alpha + \mathrm{Conv}(\mathfrak{S}_n \beta)$$

In the second case the condition is not satisfied. There are cancellations and the situation is more complicated. We can only say

$$\mathrm{Horn}(\alpha, \beta) \subset \alpha + \mathrm{Conv}(\mathfrak{S}_n \beta).$$

2. Horn's problem for pseudo-Hermitian matrices

We fix p, q, $p+q=n$. $\mathcal{H}_{p,q}(\mathbb{C})$ space of pseudo-Hermitian matrices: X is a complex $n \times n$ matrix, $X^* = I_{p,q} X I_{p,q}$, with

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Note that $\mathrm{Lie}\, U(p,q) = i\mathcal{H}_{p,q}(\mathbb{C})$.
Hermitian form: for $u, v \in \mathbb{C}^n$

$$[u,v] = u_1\overline{v_1} + \cdots + u_p\overline{v_p} - u_{p+1}\overline{v_{p+1}} - \cdots - u_{p+q}\overline{v_{p+q}}.$$

Convex cone Ω in $\mathcal{H}_{p,q}(\mathbb{C})$, $U(p,q)$-invariant:

$$\Omega = \{X \in \mathcal{H}_{p,q}(\mathbb{C}) \mid [Xu,u] > 0, \forall u \ s.t.\ [u,u] = 0\}.$$

Spectral Theorem
For $X \in \Omega$, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of X are real, and X is diagonalizable in the following sense:

$$X = U\mathrm{diag}(\lambda_1, \ldots, \lambda_n)U^*, \text{ with } U \in U(p, q).$$

Furthermore $\lambda_i > \lambda_j, \ (1 \leq i \leq p, \ p+1 \leq j \leq p+q)$.

Horn's problem
For two pseudo-Hermitian matrices $A, B \in \Omega$, what can be said about the eigenvalues of the sum $C = A + B$?
In this case Horn's set is unbounded.

For a diagonal matrix $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ in Ω, the orbit $\mathcal{O}_\alpha$ is contained in Ω. One says that the orbit $\mathcal{O}_\alpha$ is of convex type.
Let Proj be the orthogonal projection of $\mathcal{H}_{p,q}(\mathbb{C})$ on the space D_n of diagonal matrices.
$\mathcal{W}_0 = \mathfrak{S}_p \times \mathfrak{S}_q$ acting on D_n.
$\mathcal{C}$ is the cone in $D_n \simeq \mathbb{R}^n$ generated by the vectors $e_i - e_j$ $(1 \leq i \leq p,\ p+1 \leq j \leq p+q))$.

Non compact convexity theorem (Paneitz, 1982)

$$\mathrm{Proj}(\mathcal{O}_\alpha) = \mathrm{Conv}(\mathcal{W}_0\alpha) + \mathcal{C}.$$

This is the analogue of Horn's convexity theorem.
Note that

$$\mathrm{Conv}(\mathcal{W}_0\alpha) = \mathrm{Conv}(\mathfrak{S}_p\alpha') \times \mathrm{Conv}(\mathfrak{S}_q\alpha")$$

with $\alpha' = (\alpha_1, \ldots, \alpha_p)$, $\alpha" = (\alpha_{p+1}, \ldots, \alpha_{p+q})$.

We fix a Haar measure ω on $U(p,q)$, and define the orbital measure μ_α by

$$\int_{\mathcal{O}_\alpha} f(X)\mu_\alpha(dX) = \int_{U(p,q)} f(UAu^*)\omega(dU).$$

The measure μ_α is well defined since the isotropy subgroup of A is compact. The measure μ_α is unbounded. Therefore we cannot consider the probabilistic point of view.

Projection of the orbital measure

Let G be the measure on $\mathbb{R}^n$ defined by

$$\int_{\mathbb{R}^n} \varphi(x) G(dx) = \int_{\mathbb{R}^{pq}} \varphi\Big(\sum_{i=1}^{p} \sum_{j=p+1}^{p+q} t_{ij}(e_i - e_j)\Big) \prod_{i=1}^{p} \prod_{j=p+1}^{p+q} dt_{ij}.$$

The support of G is equal to the cone $\mathcal{C}$, and G has a density with respect to the hyperplane with equation $x_1 + \cdots + x_n = 0$, which is piecewise polynomial.

Theorem

$$\mathfrak{M}_\alpha := \mathrm{Proj}(\mu_\alpha) = C \ G * (M_{\alpha'} \times M_{\alpha''}).$$

($M_{\alpha'}$ and $M_{\alpha''}$ are Heckman's measures.)

For the proof one uses a formula established by Ben Saïd and Ørsted (2005) for the Laplace transform of the orbital measure μ_α: for $\mathrm{Re}\, Z \in \Omega$,

$$\mathcal{L}\mu_\alpha(Z) = \int_{\mathcal{O}_\alpha} e^{-\mathrm{tr}(ZX)} \mu_\alpha(dX).$$

Theorem For $Z = \mathrm{diag}(z_1, \ldots, z_n)$,

$$\mathcal{L}\mu_\alpha(Z) = \frac{C}{V_n(\alpha) V_n(z)} \det(e^{-\alpha_i z_j})_{1 \le i,j \le p} \det(e^{-\alpha_i z_j})_{p+1 \le i,j \le p+q}.$$

This is an analogue of the Harish-Chandra-Itzykson-Zuber formula. The Laplace transform, w. r. to D_n of the projection $\mathfrak{M}_\alpha$ is equal to the restriction to D_n of the Laplace transform of μ_α w. r. to $\mathcal{H}_{p,q}$. One checks that, when restricted to D_n,

$$\mathcal{L}\mu_\alpha(Z) = \widehat{G}(z) \widehat{M_{\alpha'}}(z_1, \ldots, z_p) \widehat{M_{\alpha"}}(z_{p+1}, \ldots, z_{p+q}).$$

Similarly one proves

Theorem

$$\nu_{\alpha,\beta} = C\, V_n(t)\eta_\alpha * \mathfrak{M}_\beta,$$

where

$$\eta_\alpha = \sum_{w\in\mathcal{W}_0} \varepsilon(w)\delta_{w\alpha}.$$

It follows that

$$\mathrm{Horn}(\alpha,\beta) \subset \alpha + \big(\mathrm{conv}(\mathcal{W}_0\beta) + \mathcal{C}\big).$$

In the case of the space of real symmetric matrices $Sym(n, \mathbb{R})$,
with the action of the orthogonal group $O(n)$, for $n \geq 3$,
in general we don't know any explicit formula for Heckman's measure,
and for the measures $\nu_{\alpha,\beta}$.
This setting is natural, however the problem is more difficult
than in the case of the space of Hermitian matrices,
and one should not expect any explicit formula.
The supports should be the same as in the case of $\mathcal{H}_n(\mathbb{C})$
with the action of the unitary group $U(n)$, according to Fulton (1998).
However we know the measure $\nu_{\alpha,\beta}$ in the special case of B being of rank one.

3. Horn's problem for real symmetric or Hermitian matrices, the case of B of rank one

$$\mathcal{H}_n(\mathbb{F}) = \begin{cases} Sym(n,\mathbb{R}) & \text{if } \mathbb{F}=\mathbb{R}\ , \\ Herm(n,\mathbb{C}) & \text{if } \mathbb{F}=\mathbb{C}\ . \end{cases}$$

The group $U_n(\mathbb{F})$ acts on the space $\mathcal{H}_n(\mathbb{F})$: $X \mapsto UXU^*$.

$$U_n(\mathbb{F}) = \begin{cases} O(n) & \text{if } \mathbb{F}=\mathbb{R}\ , \\ U(n) & \text{if } \mathbb{F}=\mathbb{C}\ . \end{cases}$$

$$\begin{aligned} A &= \operatorname{diag}(\alpha_1,\ldots,\alpha_n),\ \alpha_1 \geq \cdots \geq \alpha_n, \\ B &= \operatorname{diag}(\beta_1,\ldots,\beta_n),\ \beta_1 \geq \cdots \geq \beta_n. \end{aligned}$$

The eigenvalues of $Z = U_1 A U_1^* + U_2 B U_2^*$ $(U_1, U_2 \in U_n(\mathbb{F})$ are the zeros of

$$\det(zI - U_1 A U_1^* - U_2 B U_2^*) = \det(zI - A - UBU^*) = 0,$$

where $U = U_1^{-1} U_2$. With $Y = UBU^* \in \mathcal{O}_\beta$,

$$\det(zI - A - Y) = \det(zI - A) \det\bigl(I - (zI - A)^{-1} Y\bigr).$$

Consider the rational function

$$f(z) = \frac{\det(zI - A - Y)}{\det(zI - A)} = \det\bigl(I - (zI - A)^{-1} Y\bigr).$$

The eigenvalues λ_i of Z are the zeros of f, and the poles of f are the eigenvalues α_i of A.

We assume now that B is of rank one:

$$B = \mathrm{diag}(\beta_1, \ldots, \beta_n), \ \beta_1 > 0, \ \beta_2 = \cdots = \beta_n = 0.$$

Recall that, if the rank of the matrix T is equal to one, then

$$\det(I + T) = 1 + \mathrm{tr}(T).$$

The matrix $Y = UBU^*$ is of rank one, and the matrix $(zI - A)^{-1}Y$ as well. Therefore

$$f(z) = 1 - \mathrm{tr}\big((zI - A)^{-1}Y\big) = 1 - \sum_{i=1}^{n} \frac{Y_{ii}}{z - \alpha_i},$$

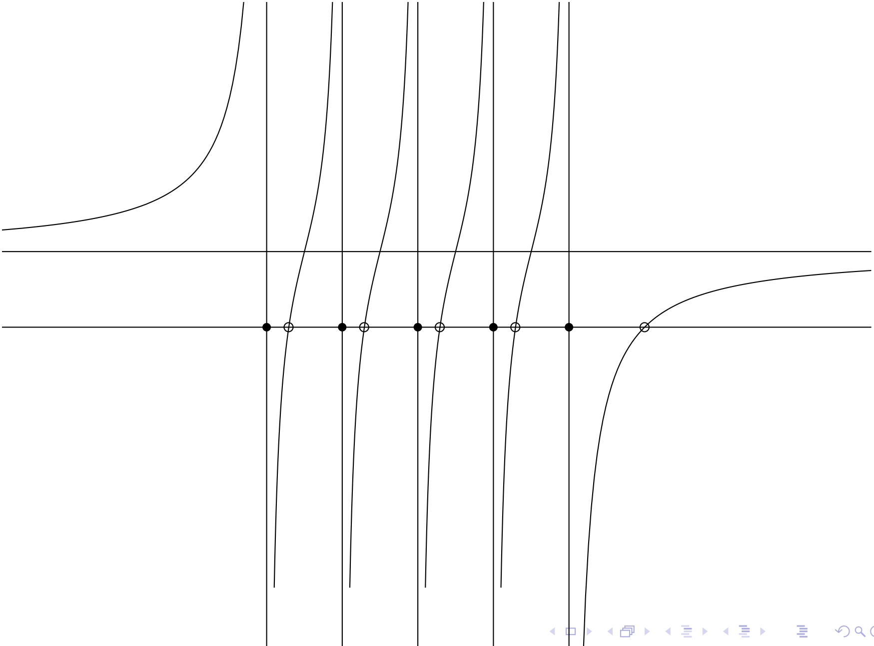
with $Y_{ii} = b|U_{i1}|^2 \geq 0$.

Interlacing property

Theorem (Fromkin-Goldberger, 2006) The set $\mathrm{Horn}(\alpha, \beta)$ of possible eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ of the sum $Z = X + Y$, with $X \in \mathcal{O}_\alpha$, $Y \in \mathcal{O}_\beta$, is determined by

$$\lambda_1 \geq \alpha_1 \geq \lambda_2 \geq \alpha_2 \geq \cdots \geq \lambda_n \geq \alpha_n,$$
$$\lambda_1 + \cdots + \lambda_n = \alpha_1 + \cdots + \alpha_n + b.$$

This last equality means simply $\mathrm{tr}(Z) = \mathrm{tr}(X) + \mathrm{tr}(Y)$.



Heckman's measure

For simplicity we assume $b = 1$: $\beta = (1, 0, \dots, 0)$. The orbit $\mathcal{O}_\beta$ is the set of the matrices

$$x = (u_i \overline{u_j}) \quad (u \in S(\mathbb{F}^n), \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}).$$

The image of $\mathcal{O}_\beta$ under the projection q of $\mathcal{H}_n(\mathbb{F})$ on D_n given by

$$x = (u_i \overline{u_j}) \mapsto (|u_1|^2, \dots, |u_n|^2)$$

is the simplex

$$\Sigma = \{w = (w_1, \dots, w_n) \in \mathbb{R}^n \mid w_i \geq 0,\ w_1 + \cdots + w_n = 1\}$$

Recall that Heckman's measure is the image under the projection Proj of the orbital measure $\mathcal{O}_\beta$. In the present case it is a Dirichlet distribution

$$\int_\Sigma f(w) D_\theta(dw) = \frac{1}{C_n(\theta)} \int_\Sigma f(w)(w_1 w_2 \dots w_n)^{\theta-1} dw_1 \wedge \dots \wedge dw_{n-1},$$

where

$$\theta = \frac{1}{2}\mathrm{dim}_{\mathbb{R}}\mathbb{F} = \frac{1}{2} \text{ or } 1, \; C_n(\theta) = \frac{\Gamma(\theta)^n}{\Gamma(n\theta)}.$$

We saw that the eigenvalues are the roots of the equation

$$f(z) := 1 - \sum_{i=1}^{n} \frac{w_i}{z - \alpha_i} = 0.$$

We know the joint distribution of the random variables $w_1, \ldots, w_n$: this is the Dirichlet distribution D_θ. The equation above defines implicitely a map

$$(\lambda_1, \ldots, \lambda_n) \mapsto (w_1, \ldots, w_n).$$

It is possible to compute the Jacobian of this map by using the following Cauchy identity

$$\det\Big(\frac{1}{\lambda_j - \alpha_i}\Big) = V_n(\lambda_1, \ldots, \lambda_n) V_n(\alpha_1, \ldots, \alpha_n) \prod_{i,j=1}^{n} \frac{1}{\lambda_j - \alpha_i}.$$

Theorem (Forrester-Zhang,2019) The joint distribution $\nu^+_{\alpha,\beta}$ of the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ of the sum $Z = X + Y$ is given by

$$\int f(\lambda)\nu^+_{\alpha,\beta}(d\lambda) = \frac{1}{C_n(\theta)} \frac{1}{V_n(\alpha_1, \ldots, \alpha_n)^{2\theta-1}} \int_{\mathrm{Horn}(\alpha,\beta)} \prod_{i,j=1}^{n} |\lambda_j - \alpha_i|^{\theta-1} V_n(\lambda_1, \ldots, \lambda_n) d\lambda_1 \wedge \cdots \wedge d\lambda_{n-1}$$

Recall that $\mathrm{Horn}(\alpha, \beta)$ is equal to the part of the hyperplane

$$\lambda_1 + \cdots + \lambda_n = \alpha_1 + \cdots + \alpha_n + 1$$

defined by the interlacing property

$$\lambda_1 \geq \alpha_1 \geq \lambda_2 \geq \alpha_2 \geq \cdots \geq \lambda_n \geq \alpha_n.$$

More generally one could consider Horn's problem
for the adjoint action of a compact Lie group on its Lie algebra.
The Fourier transform of an orbital measure is explicitely given
by the Harish-Chandra integral formula [1957].
Heckman's paper [1982] is written in this framework. One can expect
that there is an analogue of our result in this setting.
In particular one can consider the action of the orthogonal group
on the space of real skew-symmetric matrices, as Zuber did (2017).

4. Relation to representation theory

π_λ irreducible representation of $U(n)$ with highest weight λ,
$\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ $(\lambda_i \in \mathbb{Z})$.

Littlewood-Richardson coefficients $c_{\alpha,\beta}^{\gamma}$:

$$\pi_\alpha \otimes \pi_\beta = \sum_\gamma c_{\alpha,\beta}^{\gamma} \pi_\gamma.$$

Theorem $c_{\alpha,\beta}^{\gamma} \neq 0$ if and only if $\gamma \in \mathrm{Horn}(\alpha, \beta)$;
i.e. there exist $n \times n$ Hermitian matrices A, B, C with $C = A + B$, the α_i are the eigenvalues of A, the β_i of B, the γ_i of C.

(Klyachko, 1998; Knutson, Tao, 1999)

A recent result by Coquereaux, McSwiggen, and Zuber.
(arXiv, 2019)

Following Berenstein and Zelevinsky (1992), for a triple $(\alpha, \beta, \gamma) \in \mathbb{R}^{3n}$ they consider a polytope $H_{\alpha,\beta}^{\gamma}$ in $\mathbb{R}^N$. By a result of Berenstein-Zelevinsky, for a triple (λ, μ, ν),

$$c_{\lambda,\mu}^{\nu} = \#\{\text{integer points in } H_{\lambda,\mu}^{\nu}\}.$$

Coquereaux, McSwiggen and Zuber prove that the probability density function for the Horn's problem is given by

$$p(\alpha, \beta; \gamma) = C \frac{V_n(\gamma)}{V_n(\alpha) V_n(\beta)} \mathrm{vol}(H_{\alpha,\beta}^{\gamma}).$$

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