Probability Measures of Representation Theoretic Origin

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Plan of the talk

1. Basic example: Plancherel measures & Ulam's problem

2. The BC type Z-measures

2.1. Orthogonal Z-measures & representation theory

2.2. The BC type Z-measures

3. q-analogues of rep. theoretic measures

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1. **Basic example: Plancherel measures & Ulam's problem**
2. **The BC type Z-measures**
    2.1. Orthogonal Z-measures & representation theory
    2.2. The BC type Z-measures
3. **q-analogues of rep. theoretic measures**

1. A brief reminder on partitions

A **partition** is an integer sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots)$ with $\lambda_k = 0$ for large enough k. The *size* of λ is $|\lambda| := \sum_i \lambda_i$.

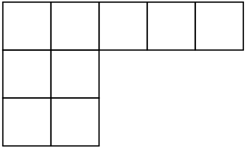

Figure: Partition (5, 2, 2, 0, 0, . . .)

In representation theory, partitions λ of size N parametrize the irreducible representations V_λ of the symmetric group $\mathfrak{S}_N$.

1. Plancherel measures

The **Plancherel measure** of level N is

$$P_N(\lambda) := \frac{(\dim \lambda)^2}{N!}, \quad |\lambda| = N,$$

where $\dim \lambda = \dim(V_\lambda)$.

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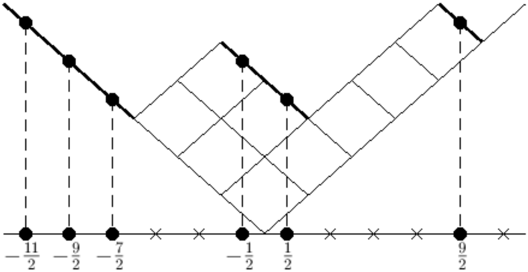
where $\dim \lambda = \dim(V_\lambda)$.

Identify partitions with subsets (point configurations) of $\mathbb{Z}' := \{\cdots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots\}$:

$$\lambda \mapsto X(\lambda) := \left\{ \dots,\ \lambda_3 - \frac{5}{2},\ \lambda_2 - \frac{3}{2},\ \lambda_1 - \frac{1}{2} \right\}.$$

1. Plancherel measures



$$N = 9, \quad \lambda = (5,\ 2,\ 2,\ 0,\ 0,\ \cdots),$$

$$\Rightarrow X(\lambda) = \left\{\ldots,\ 0 - \frac{9}{2},\ 0 - \frac{7}{2},\ 2 - \frac{5}{2},\ 2 - \frac{3}{2},\ 5 - \frac{1}{2}\right\}.$$

1. Limit of Plancherel measures

To study the rightmost particles, consider the embeddings:

$$\begin{aligned} \mathfrak{i}_N : \{\lambda : |\lambda| = N\} &\hookrightarrow \mathrm{Conf}(\mathbb{R}) \\ \lambda &\mapsto \frac{1}{N^{1/6}}\left(X(\lambda) - 2\sqrt{N}\right) \end{aligned}$$

Let $\widetilde{P}_N$ be the pushforward of P_N under the map $\mathfrak{i}_N$.

1. Limit of Plancherel measures

Theorem(Baik-Deift-Johansson'99, Borodin-Okounkov-Olshanski'00)

The weak limit $P := \lim_{N\to\infty} \widetilde{P}_N$ exists.

For $k \geq 1$, the ***k*^th^ point correlation function** of P is

$$\rho_k(x_1, x_2, \ldots, x_k) = \det[K(x_i, x_j)]_{i,j=1}^{k},$$

$$K(x,y) = \frac{A(x)A'(y) - A'(x)A(y)}{x-y},$$

where $\mathrm{A}(x) := \frac{1}{\pi}\int_0^\infty \cos(\frac{u^3}{3} + xu)du$ is the *Airy function*.

1. Ulam's problem

Ulam's problem:

Let π_N be a uniform random permutation of $(1, 2, \ldots, N)$.

Let $L(\pi_N) =$ length of longest increasing subsequence of π_N.

How does $L(\pi_N)$ behave as $N \to \infty$?

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How does $L(\pi_N)$ behave as $N \to \infty$?

Key observation:

$L(\pi_N) \overset{\mathrm{d}}{=} \lambda_1$ if λ is Plancherel(N)-distributed.

Conclusion:

$$F(s) = \lim_{N \to \infty} \mathrm{Prob}\left(\frac{L(\pi_N) - 2\sqrt{N}}{N^{1/6}} \leq s \right)$$

exists for all $s \in \mathbb{R}$, is expressed in terms of Airy functions, etc.

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2. **The BC type Z-measures**
   2.1. Orthogonal Z-measures & representation theory
   2.2. The BC type Z-measures

3. **q-analogues of rep. theoretic measures**

2.1. Orthogonal Z-measures

Let $\mathbb{Y}_N := \{\lambda : \lambda_{N+1} = 0\}$.
(These partitions parametrize certain irreps. of $SO(2N+1)$.)

Let $\mathbf{z}, \mathbf{z}' \in \mathbb{C}$ satisfy certain constraints.

2.1. Orthogonal Z-measures

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Let $\mathbf{z}, \mathbf{z}' \in \mathbb{C}$ satisfy certain constraints.

The **orthogonal Z-measure** of level N is

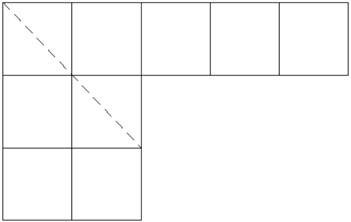
$$P^{SO}_{N|\mathbf{z},\mathbf{z}'}(\lambda) := \frac{1}{Z^{SO}_{N|\mathbf{z},\mathbf{z}'}} Dim_N(\lambda)^2 \prod_{k=1}^{N} w^{SO}_{N|\mathbf{z},\mathbf{z}'}(\widetilde{\lambda}_k), \quad \lambda \in \mathbb{Y}_N,$$

where $\widetilde{\lambda}_k := \lambda_k + N - k + \frac{1}{2}$, and

$$w^{SO}_{N|\mathbf{z},\mathbf{z}'}(x) := \frac{1}{\Gamma(\mathbf{z} - x + N)\Gamma(\mathbf{z}' - x + N)\Gamma(\mathbf{z} + x + N)\Gamma(\mathbf{z}' + x + N)}.$$

2.1. Limit of orthogonal Z-measures

Identify $\mathbb{Y}_N$ with point configurations on $\mathbb{R}_+^* := (0, +\infty) \setminus \{1\}$:



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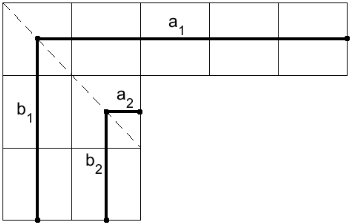

$$\mathfrak{i}_N : \mathbb{Y}_N \hookrightarrow \mathrm{Conf}(\mathbb{R}_+^*)$$

$$\lambda \mapsto \left\{1 + \frac{a_i}{N}\right\}_{i=1}^{d} \sqcup \left\{1 - \frac{b_j}{N}\right\}_{j=1}^{d}$$

and let $\widetilde{P}^{SO}_{N|\mathbf{z},\mathbf{z}'}$ be the corresponding measure on $\mathrm{Conf}(\mathfrak{X})$.

2.1. Limit of orthogonal Z-measures

Theorem (Cuenca '18).
The weak limit $P_{\mathbf{z},\mathbf{z}'} := \lim_{N\to\infty} \widetilde{P}^{SO}_{N|\mathbf{z},\mathbf{z}'}$ exists.

For $k \geq 1$, the k^{th} point correlation function of $P_{\mathbf{z},\mathbf{z}'}$ is

$$\rho_k(x_1, x_2, \ldots, x_k) = \det[K_{\mathbf{z},\mathbf{z}'}(x_i, x_j)]_{i,j=1}^k,$$

$$K_{\mathbf{z},\mathbf{z}'}(x, y) = \frac{R(x)S(y) - S(x)R(y)}{x - y},$$

where $R(x) = R_{\mathbf{z},\mathbf{z}'}(x)$, $S(x) = S_{\mathbf{z},\mathbf{z}'}(x)$ are given explicitly....

2.1. Limit of orthogonal Z-measures

....for example, if $x > 1$, then

$$R(x) = \frac{\sqrt{\sin \pi \mathbf{z} \ \sin \pi \mathbf{z}'}}{\sqrt{2}\pi} \cdot x^{\frac{1}{4}-\mathbf{z}'}(x-1)^{\frac{\mathbf{z}'-\mathbf{z}}{2}} \cdot {}_2F_1\left[\begin{matrix} \mathbf{z}'-\frac{1}{2} & \mathbf{z}' \\ \mathbf{z}+\mathbf{z}'-\frac{1}{2} & \end{matrix};\ \frac{1}{x}\right],$$

$$S(x) = \frac{\sqrt{2\sin \pi \mathbf{z} \ \sin \pi \mathbf{z}'}}{\pi} \cdot \frac{\Gamma(\mathbf{z}+\frac{1}{2})\Gamma(\mathbf{z}'+\frac{1}{2})\Gamma(\mathbf{z}+1)\Gamma(\mathbf{z}'+1)}{\Gamma(\mathbf{z}+\mathbf{z}'+\frac{1}{2})\Gamma(\mathbf{z}+\mathbf{z}'+\frac{3}{2})}$$

$$\cdot x^{-\frac{3}{4}-\mathbf{z}'}(x-1)^{\frac{\mathbf{z}'-\mathbf{z}}{2}} \cdot {}_2F_1\left[\begin{matrix} \mathbf{z}'+\frac{1}{2} & \mathbf{z}'+1 \\ \mathbf{z}+\mathbf{z}'+\frac{3}{2} & \end{matrix};\ \frac{1}{x}\right],$$

where ${}_2F_1\left[\begin{smallmatrix} a & b \\ c & \end{smallmatrix}; z\right] = \sum_{n=0}^{\infty} \frac{z^n}{n!} \prod_{i=1}^{n} \frac{(a+i-1)(b+i-1)}{(c+i-1)}$ is Gauss's hypergeometric function.

2.1. Representation-theoretic interpretation

Spherical characters $\Psi : K \to \mathbb{C}$:

- central functions: $\Psi(xyx^{-1}) = \Psi(y)$
- positive-definite: $[\Psi(g_j^{-1} g_i)]_{i,j=1,\dots,n}$ is Hermitian and ≥ 0
- normalized: $\Psi(e) = 1$

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Example: When $K = SO(2N+1)$,

$$\Psi = \sum_\lambda c_\lambda \frac{\chi_\lambda}{\mathrm{Dim}_N(\lambda)}, \quad \sum_\lambda c_\lambda = 1, \ c_\lambda \geq 0.$$

2.1. Representation-theoretic interpretation

Spherical representations T of $K \subset G$:

Hilbert space H, continuous $T : G \to U(H)$, unit vector $v \in H$ such that $T(K)v = v$ and $\overline{T(G)v} = H$.

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Example: For

$$G = SO(2N+1) \times SO(2N+1),$$
$$K = SO(2N+1) \subset G \quad (\text{via } x \mapsto (x, x)),$$

an spherical representation is

$$H_N = L^2(SO(2N+1), \mu_N),$$
$$(T(g_1, g_2)f)(x) = f(g_2^{-1} x g_1),$$
$$v_N = \text{normalized constant.}$$

2.1. Representation-theoretic interpretation

For

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the notions of characters and representations are **equivalent**! For example, given representation T, its character is

$$\Psi(X) = (T(X, 1)v, v)_H, \quad X \in SO(\infty).$$

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Examples of representations? Olshanski found a distinguished family of spherical representations $\{T_{\mathbf{z}}\}_{\mathbf{z}}$:

$$\cdots \hookrightarrow H_{N-1} \hookrightarrow H_N \hookrightarrow \ldots H = \lim_{\to} H_N$$
$$\cdots \hookrightarrow v_{N-1} \hookrightarrow v_N \hookrightarrow \ldots v_\infty = \lim_{\to} v_N$$

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Examples of characters? Olshanski found a distinguished family of spherical characters $\{\Psi_{\mathbf{z}}\}_{\mathbf{z}}$:

$$\Psi_{\mathbf{z}} : SO(\infty) \to \mathbb{C} \text{ is the unique map such that}$$
$$\Psi_{\mathbf{z}}|_{SO(2N+1)} := \sum_{\lambda} \frac{\chi_\lambda}{\mathrm{Dim}_N(\lambda)} \mathbf{P}^{\mathbf{SO}}_{\mathbf{N}|\mathbf{z},\bar{\mathbf{z}}}(\boldsymbol{\lambda}).$$

2.1. Representation-theoretic interpretation

Semisimplicity. If Υ is the space of irreducible characters ψ^ω, any character Ψ equals

$$\Psi(g) = \int_\Upsilon \psi^\omega(g) P(d\omega), \quad g \in SO(\infty),$$

for a unique prob. measure P on Υ (**the spectral measure**).

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Can one describe the measure $P_{\mathbf{z}}$ corresponding to $\Psi_{\mathbf{z}}$? This is the **problem of noncommutative harmonic analysis**.

2.1. Representation-theoretic interpretation

Characterization of irreps. The space Υ was characterized for $SO(\infty)$ (Boyer '92, Okounkov-Olshanski '06):

$$\Upsilon \cong \Big\{(\alpha,\beta,\delta) \in \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+ \ : \ \alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \\ 1 \geq \beta_1 \geq \beta_2 \geq \cdots \geq 0, \ \delta \geq \sum_{i=1}^{\infty} (\alpha_i + \beta_i)\Big\}$$

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A general framework shows that, after the embedding

$$\mathfrak{i} : \Upsilon \hookrightarrow \mathrm{Conf}(\mathbb{R}_+^*)$$
$$(\alpha, \beta, \delta) \mapsto (\{1 + \alpha_i\}_{i \geq 1} \sqcup \{1 - \beta_j\}_{j \geq 1}) \setminus \{0, 1\},$$

the image of $P_{\mathbf{z}}$ is equal to $\lim_{N \to \infty} P^{SO}_{N|\mathbf{z}, \bar{\mathbf{z}}}$.

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the image of $P_{\mathbf{z}}$ is equal to $\lim_{N\to\infty} P^{SO}_{N|\mathbf{z},\bar{\mathbf{z}}}$.

Conclusion: To describe the spectral measure $P_{\mathbf{z}}$, we had to compute the weak limit: $\lim_{N\to\infty} P^{SO}_{N|\mathbf{z},\bar{\mathbf{z}}}$!!!

Plan of the talk

2.2. BC type Z-measures

The problem of harmonic analysis can also be formulated for $Sp(\infty) = \lim_{\to} Sp(N)$. The answer is very similar.

Is there a unifying theory for $SO(\infty)$, $Sp(\infty)$, and other inductive limit groups?

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Is there a unifying theory for $SO(\infty)$, $Sp(\infty)$, and other inductive limit groups?

Observation #1: The character $\chi_\lambda^{SO(2N+1)}(M)$ is an explicit function of the eigenvalues $\{1, \pm x_1, \ldots, \pm x_N\}$ of M:

$$\boxed{\chi_\lambda^{SO(2N+1)}(M) = \mathfrak{J}_\lambda^{a,b}(x_1, \ldots, x_N)\Big|_{(a,b)=\frac{1}{2},-\frac{1}{2}}}$$

where $\mathfrak{J}_\lambda^{a,b}(x_1, \ldots, x_N)$ are the multivariate Jacobi polynomials.

2.2. BC type Z-measures

Observation #2: The measures $\{P_N\}_{N\geq 1}$ come from Fourier expansions of the restrictions $\Psi_z|_{SO(2N+1)}$. This implies a **coherence relation** between P_N and P_{N-1}:

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Define $\boldsymbol{C^N_{N-1}}(\lambda,\mu)$ via the branching

$$\left.\frac{\chi_\lambda^{SO(2N+1)}}{Dim_N(\lambda)}\right|_{SO(2N-1)} = \sum_{\mu\in\mathbb{Y}_{N-1}} \boldsymbol{C^N_{N-1}}(\lambda,\mu)\frac{\chi_\mu^{SO(2N-1)}}{Dim_{N-1}(\mu)}, \quad \lambda\in\mathbb{Y}_N.$$

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The coherence relations are:

$$\boxed{\sum_{\lambda\in\mathbb{Y}_N} P_N(\lambda)\boldsymbol{C^N_{N-1}}(\lambda,\mu) = P_{N-1}(\mu), \quad \mu \in \mathbb{Y}_{N-1}.}$$

2.2. BC type Z-measures

Miraculously, the orthogonal Z-measures admit 2-parameter (a, b) generalizations (Olshanski-Osinenko '12):

- $$\frac{\mathfrak{J}_\lambda^{a,b}(x_1, \dots, x_{N-1}, 1)}{\mathfrak{J}_\lambda^{a,b}(1, \dots, 1, 1)} = \sum_\mu \boldsymbol{C_{N-1}^N}(\lambda, \mu) \frac{\mathfrak{J}_\mu^{a,b}(x_1, \dots, x_{N-1})}{\mathfrak{J}_\mu^{a,b}(1, \dots, 1)}.$$

- The coherent measures are

$$P_{N|z,z',a,b}(\lambda) \propto \prod_{1 \le i < j \le N} (\widetilde{\lambda}_i - \widetilde{\lambda}_j)^2 \prod_{k=1}^{N} w_{z,z',a,b}(\widetilde{\lambda}_k),$$

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- The coherent measures are
$$P_{N|z,z',a,b}(\lambda) \propto \prod_{1\le i<j\le N} (\widetilde{\lambda}_i - \widetilde{\lambda}_j)^2 \prod_{k=1}^{N} w_{z,z',a,b}(\widetilde{\lambda}_k),$$
$w_{z,z',a,b}(x)$ is the weight for Racah polynomials.
- Theorem (C '18) The limit $P_{z,z',a,b} = \lim_{N\to\infty} P_{N|z,z',a,b}$ has determinantal correlation functions.

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3. **q-analogues of rep. theoretic measures**

3. q-analogues of rep-theoretic measures

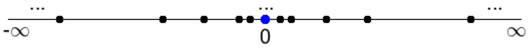
Wishful thinking: Many of our objects have q-analogues:

$$SO(N), Sp(N) \mapsto SO_q(N), Sp_q(N)$$
$$\text{Jacobi orthogonal polys} \mapsto \text{little } q\text{-Jacobi orthogonal polys}$$
$$\text{hypergeometric functions } {}_2F_1 \mapsto q\text{-hypergeometric functions } {}_2\phi_1$$

Question: Are there *q-analogues* of representation theoretic measures?

3. q-analogues of rep-theoretic measures

Answer: Yes!! Point configurations live in the 2-sided lattice $\mathfrak{L} = \zeta_-\mathbf{q}^{\mathbb{Z}} \sqcup \zeta_+\mathbf{q}^{\mathbb{Z}}$, where $\zeta_- < 0 < \zeta_+,\ 0 < \mathbf{q} < 1$:



i.e. $\Upsilon_N =$ set of size N subsets of $\mathfrak{L}$.

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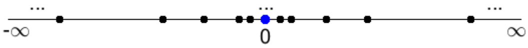


i.e. Υ_N = set of size N subsets of $\mathfrak{L}$.

We seek:

(1) $\boldsymbol{C^N_{N-1}}(X, Y)$, where $X \in \Upsilon_N$, $Y \in \Upsilon_{N-1}$.

(2) Sequences of probability measures $\{P_N(X)\}_{N \geq 1}$ satisfying:

$$\sum_{X \in \Upsilon_N} P_N(X) \boldsymbol{C^N_{N-1}}(X, Y) = P_{N-1}(Y).$$

3. q-analogues of rep-theoretic measures

Example 1: Gorin-Olshanski ('16) studied the q-ZW measures (q-analogues of measures related to $U(\infty)$):

(1)

$$\boldsymbol{C^N_{N-1}}(X, Y) = \mathbf{1}_{\{Y \prec X\}} \prod_{y \in Y} |y| \cdot (1-q) \cdots (1-q^{N-1}) \cdot \frac{V(Y)}{V(X)}.$$

(Come from branching of q-shifts of Schur polynomials)

(2)

$$P_N(X) = \frac{1}{Z_N} V(X)^2 \prod_{x \in X} w_{\mathbf{q};\alpha,\beta,\gamma^N,\delta^N}(x),$$

$w_{\mathbf{q};\alpha,\beta,\gamma,\delta}(x)$ is the weight for big q-Jacobi polynomials.

3. q-analogues of rep-theoretic measures

Example 2: Cuenca-Olshanski ('18, '19+) studied q-Z measures (q-analogues of measures related to $SO(\infty), Sp(\infty)$):

(1)

$$\boldsymbol{C^N_{N-1}}(X, Y) = \mathbf{1}_{\{Y \prec X\}} \cdot (\text{certain function of } Y) \cdot \frac{V(Y)}{V(X)}.$$

(Come from branching of little q-Jacobi polynomials)

(2)

$$P_N(X) = \frac{1}{Z_N} V(X)^2 \prod_{x \in X} w_{\mathbf{q}; s_0^N, s_1^N, s_2^N, s_3^N}(x),$$

$w_{\mathbf{q}; s_0, s_1, s_2, s_3}(x)$ is the weight for q-Racah polynomials.

3. q-analogues of rep-theoretic measures

Several basic properties of the q-ZW and q-Z measures have been proved in Gorin-Olshanski('16), C-Olshanski ('19+) and C-Gorin-Olshanski ('19), e.g:

Theorem. The q-ZW measures $P_{\mathbf{q};\alpha,\beta,\gamma,\delta}$ and q-Z measures $P_{\mathbf{q};s_0,s_1,s_2,s_3}$ (obtained as limits $\lim_{N\to\infty} P_N$) have determinantal correlation functions, in terms of the basic hypergeometric functions ${}_2\phi_1$, ${}_3\phi_2$.

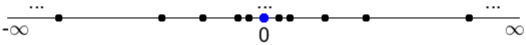
$$\left(\text{e.g. } {}_2\phi_1\begin{bmatrix} a & b \\ & c \end{bmatrix}; q; z\right] = \sum_{n=0}^{\infty} z^n \prod_{i=1}^{n} \frac{(1-aq^{i-1})(1-bq^{i-1})}{(1-cq^{i-1})(1-q^i)} \text{ is the}$$

$$q\text{-analogue of Gauss's hypergeometric function } {}_2F_1\begin{bmatrix} a & b \\ & c \end{bmatrix}; z\Bigg]\Bigg)$$

3. q-analogues of rep-theoretic measures

But even the most basic questions can't be answered yet!

- Are there limits $q \to 1$ that turn $P_{\mathbf{q};\alpha,\beta,\gamma,\delta}$, $P_{\mathbf{q};s_0,s_1,s_2,s_3}$ into the ZW measures and BC type Z measures?
- Meaning of the two-sided lattice $\mathfrak{L} = \mathfrak{L}_- \sqcup \mathfrak{L}_+$?
  Conjecturally, $\mathfrak{L}_-/\mathfrak{L}_+$ relate to rows/columns of partitions



- Interpretation from harmonic analysis/quantum groups?

Plancherel measures $\longleftarrow$ Ulam's problem

orthogonal Z measures $\longleftarrow$ harmonic analysis for $SO(\infty)$

q-Z measures $\longleftarrow$???

3. q-analogues of rep-theoretic measures

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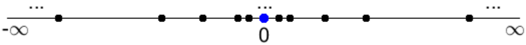

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Thank you for your attention!