Theta Correspondence for Dummies

(Correspondance Theta pour les nuls)

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$Mp = \widetilde{Sp}(2n, F)$: metaplectic group

(G, G') a reductive dual pair:

$G = \mathrm{Cent}_{Mp}(G'), \quad G' = \mathrm{Cent}_{Mp}(G)$

ψ character of F, $\rightarrow$ oscillator representation $\omega = \omega_\psi$

Definition: $\pi \in \widehat{G}, \pi' \in \widehat{G'}$, say $\pi \longleftrightarrow \pi'$ if

$$\mathrm{Hom}_{G\times G'}(\omega, \pi \boxtimes \pi') \neq 0$$

Howe Duality Theorem (Howe, Waldspurger, Gan-Takeda)
F local

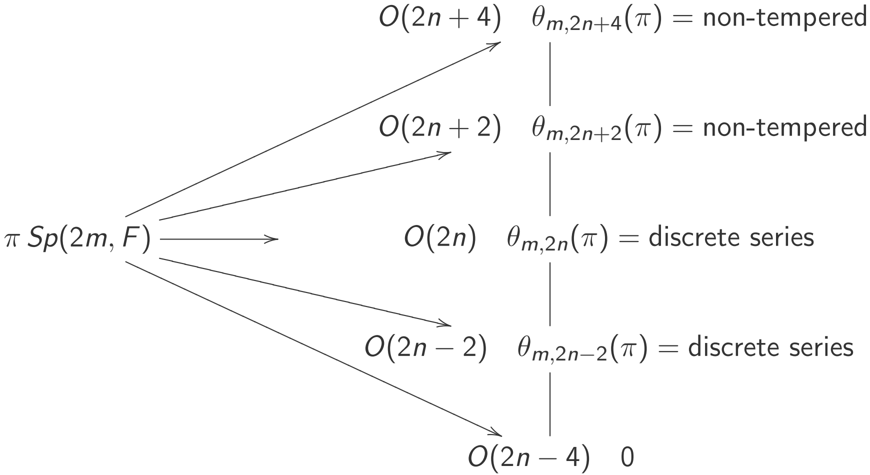
$$\pi \longleftrightarrow \pi' \text{ is a bijection}$$

(between subsets of $\widehat{G}$ and $\widehat{G}'$)

Definition: $\pi' = \theta(\pi)$, $\pi = \theta(\pi')$

Computing $\theta(\pi)$

Describe $\pi \to \theta(\pi)$ (in terms of some kinds of parameters)
Properties of the map: preserving tempered, unitary, relation on wave front sets, functoriality (Langlands/Arthur)...
Typically there are some easy cases, and some hard ones



$\Theta(\pi)$

$\theta(\pi)$ irreducible, $\omega \to \pi \boxtimes \theta(\pi)$

Defintion (Howe) $\omega(\pi)$=the maximal π-isotypic quotient of ω

$\Theta(\pi)$ ("big-theta" of π):

$$\omega(\pi) \simeq \pi \boxtimes \Theta(\pi)$$

Proof of the duality theorem: $\theta(\pi)$ is the unique irreducible quotient of $\Theta(\pi)$

Generically, $\Theta(\pi)$ is irreducible and $\theta(\pi) = \Theta(\pi)$

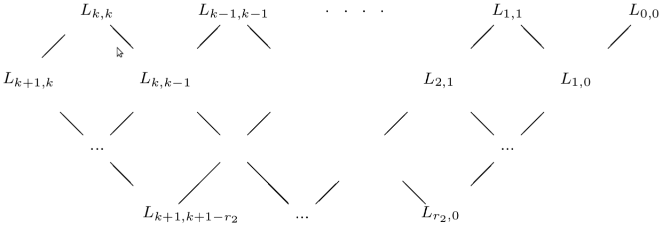
THE STRUCTURE OF $\Theta(\pi)$

$\Theta(\pi)$ is important, interesting, complicated

$\Theta(1)$ (Kudla, Rallis, ...)

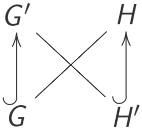
Structure of reducible principal series (Howe...)

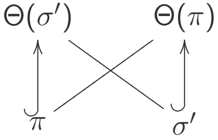
Lee/Zhu: $Sp(2n,\mathbb{R})$:



Example: see-saw pairs and reciprocity

Howe





$$\Theta(\sigma')[\pi] \boxtimes \sigma' \simeq \pi \boxtimes \Theta(\pi)[\sigma']$$

Roughly:

$$\mathrm{mult}_G(\pi, \Theta(\sigma')) = \mathrm{mult}_{H'}(\sigma', \Theta(\pi))$$

Theta correspondence and induction

$$\begin{array}{ccc} & & \mathrm{GL}(n+r) \\ & \overset{\Theta_{m,n+r}}{\nearrow} & | \\ \mathrm{GL}(m) & \xrightarrow{\Theta_{m,n}} & \mathrm{GL}(n) \end{array}$$

$m = n$: $\Theta_{n,n}(\pi) = \theta_{n,n}(\pi) = \pi^*$

Kudla: $P = MN, \quad M = GL(n) \times GL(r)$

$$\mathrm{Hom}_{\mathrm{GL}(m)}(\omega_{m,n+r}, \pi \boxtimes \mathrm{Ind}_P^{\mathrm{GL}(n+r)}(\theta_{m,n}(\pi) \otimes 1)) \neq 0$$

$$\begin{array}{ccc} & & \mathrm{Ind}_P^{GL(n+r)}(\theta_{m,n}(\pi) \otimes 1) \\ & \overset{\Theta_{m,n+r}}{\nearrow} & | \\ \pi & \xrightarrow{\theta_{m,n}} & \theta_n(\pi) \end{array}$$

Yale Freshman graduate student's dream

$\Theta_{m,n+r}(\pi) \stackrel{?}{=} \mathrm{Ind}_P^{GL(n+r)}(1 \otimes \theta_{m,n}(\pi))$

$\theta_{m,n+r}(\pi)$ is (?) the unique irreducible quotient of $\mathrm{Ind}_P^{GL(n+r)}(1 \otimes \theta_{m,n}(\pi))$

Neither is true in general

$\omega = \mathcal{S}(M_{m,n})$

filtration: ω_k: functions supported on matrices of rank $\geq k$:

$$0 = \omega_t \subset \omega_{t-1} \subset \cdots \subset \omega_0 = \omega$$

Serious issues with extensions here. . . also reducibility of induced representations

Basic Principle

$$\mathrm{Hom} \to \mathrm{Ext} \to \mathrm{EP} = \sum_i (-1)^i \mathrm{Ext}^i$$

(+ vanishing...)

Problem: Study

$\mathrm{Ext}^i_{G\times G'}(\omega, \pi \boxtimes \pi')$, $\mathrm{EP}_{G\times G'}(\omega, \pi \boxtimes \pi')$

alternatively:

$\mathrm{Ext}^i_G(\omega, \pi), \mathrm{EP}_G(\omega, \pi)$ as (virtual) representations of G'

Idea: $\mathrm{EP}_G(\omega, \pi)$ is like $\mathrm{Hom}_G(\omega, \pi)$ with everything made completely reducible... all "boundary terms" vanish

Example: GL(1), or Tate's Thesis

$(G, G') = (GL(1), GL(1)) \subset SL(2, F)$

ω: $\mathcal{S}(F)$ ($\mathcal{S} = C_c^\infty$, the Schwarz space)

$\omega(g, h)(f)(x) = f(g^{-1}xh)$ (up to $|\det|^{\pm\frac{1}{2}}$)

χ character of $GL(1)$

Question: $\mathrm{Hom}_{\mathrm{GL}(1)}(\mathcal{S}(F), \chi) =?$

$$0 \to \mathcal{S}(F^\times) \to \mathcal{S}(F) \to \mathbb{C} \to 0$$

$\mathrm{Hom}(\ ,\chi) = \mathrm{Hom}_{\mathrm{GL}(1)}(\ ,\chi)$

$$0 \to \mathrm{Hom}(\mathbb{C}, \chi) \to \mathrm{Hom}(\mathcal{S}(F), \chi) \to \mathrm{Hom}(\mathcal{S}(F^\times), \chi) \to \mathrm{Ext}(\mathbb{C}, \chi)$$

EXAMPLE: GL(1), OR TATE'S THESIS

$$0 \to \mathrm{Hom}(\mathbb{C},\chi) \to \mathrm{Hom}(\mathcal{S}(F),\chi) \to \mathrm{Hom}(\mathcal{S}(F^\times),\chi) \to \mathrm{Ext}(\mathbb{C},\chi)$$

$\chi \neq 1$:

$$0 \to 0 \to \mathrm{Hom}(\mathcal{S}(F),\chi) \to \mathrm{Hom}(\mathcal{S}(F^\times),\chi) \to 0$$

$\mathrm{Hom}_{\mathrm{GL}(1)}(\mathcal{S}(F),\chi) = \mathrm{Hom}_{\mathrm{GL}(1)}(\mathcal{S}(F^*),\chi) = \mathbb{C}$

$\chi = 1$:

$$0 \to \mathbb{C} \to \mathrm{Hom}(\mathcal{S}(F),\chi) \to \mathrm{Hom}(\mathcal{S}(F^\times),\chi) \to \mathbb{C} \to \mathrm{Ext}^1(\mathcal{S}(F),\mathbb{C}) = 0$$

$\mathrm{Hom}_{\mathrm{GL}(1)}(\mathcal{S}(F),\chi) = 1$ in all cases

Remark: Tate's thesis: this is true provided $|\chi(x)| = |x|^s$ with $s > 1$. General case: analytic continuation in χ of Tate L-functions.

Punch line:

Theorem (Adams/Prasad/Savin)
Fix m, and consider the dual pairs $(G = GL(m), GL(n))$ $n \geq 0$.
$\pi \in \widehat{G}$

$$\mathrm{EP}_G(\omega_{m,n}, \pi)^\infty \simeq \begin{cases} 0 & n < m \\ \mathrm{Ind}_P^{GL(n)}(1 \otimes \pi) & n \geq m \end{cases}$$

where $M = GL(n-m) \times GL(m)$

More details. . .

Reference: D. Prasad, *Ext Analogues of Branching Laws*

F: p-adic field, G: reductive group/F

$\mathcal{C} = \mathcal{C}_G$:category of smooth representations

$\mathcal{S}(G) = C_c^\infty(G)$, smooth compactly supported functions, smooth representation of $G \times G$

Lemma: $\mathcal{C}$ has enough projectives and injectives

$\mathrm{Ext}^i_G(X, Y)$: derived functors of $\mathrm{Hom}_G(\_, Y)$ or $\mathrm{Hom}_G(X, \_)$.

Euler-Poincare Characteristic

$P = MN \subset G$, Ind_P^G normalized induction r_P^G normalized Jacquet functor

X, Y smooth

1. $\mathrm{Ext}^i_G(X, Y) = 0$ for $i >$ split rank of G
2. $\mathcal{S}(G)$ is projective (as a left G-module)
3. $\mathrm{Hom}_G(\mathcal{S}(G), X)^{G-\infty} \simeq X$
4. $\mathrm{EP}_{\mathrm{GL}(m)}(X, Y) = 0$ (X, Y finite length)
5. $\mathrm{Ext}^i_G(X, \mathrm{Ind}_P^G(Y)) \simeq \mathrm{Ext}^i_M(r_P^G(X), Y)$
6. $\mathrm{Ext}^i_G(\mathrm{Ind}_P^G(X), Y) \simeq \mathrm{Ext}^i_M(X, r_G^{\overline{P}}(Y))$
7. Kunneth Formula (X_1 admissible):

$$\mathrm{Ext}^i_{G_1 \times G_2}(X_1 \boxtimes X_2, Y_1 \boxtimes Y_2) \simeq \bigoplus_{j+k=i} \mathrm{Ext}^j_{G_1}(X_1, Y_1) \otimes \mathrm{Ext}^k_{G_2}(X_2, Y_2)$$

X: $G \times G'$-modules (for example: ω)

Y: G-module

$\mathrm{Ext}^i_G(X,Y)$ is an G'-module (not necessarily smooth)

Definition:

$$\mathrm{Ext}^i_G(X,Y)^\infty = \mathrm{Ext}^i_G(X,Y)^{G'-\infty} \quad \text{(a smooth } G'\text{-module)}$$

Dangerous bend: $\mathrm{Ext}^i_G(X,Y)$ is (probably) not the derived functors of $Y \to \mathrm{Hom}_G(X,Y)^{G'-\infty}$

Definition: Assume $\mathrm{Ext}^i_G(X,Y)$ has finite length for all i

$\mathrm{EP}_G(X,Y) = \sum_i (-1)^i \mathrm{Ext}_G(X,Y)^\infty$ is a well-defined element of the Grothendieck group of smooth representations of G'

Back to $\Theta(\pi)$

(G, G') dual pair, ω, π irreducible representation of G

$\mathrm{EP}_G(\omega, \pi)^\infty$

$\omega \to \pi \boxtimes \Theta(\pi)$

Proposition: $\mathrm{Hom}_G(\omega, \pi)^\infty = \Theta(\pi)^\vee$

$\vee$: smooth dual

proof:

$$0 \to \omega[\pi] \to \omega \to \pi \boxtimes \Theta(\pi) \to 0$$

$\mathrm{Hom}(,\pi)$ is left exact:

$$0 \to \mathrm{Hom}_G(\pi \boxtimes \Theta(\pi), \pi) \to \mathrm{Hom}_G(\omega, \pi) \xrightarrow{\phi} \mathrm{Hom}_G(\omega[\pi], \pi)$$

$\phi = 0 \Rightarrow \mathrm{Hom}(\omega, \pi) \simeq \Theta(\pi)^*$, take the smooth vectors

Computing EP

Recall: $\omega_k = \mathcal{S}(\text{matrices of rank } \geq k)$

$$0 = \omega_t \subset \omega_{t-1} \subset \cdots \subset \omega_0 = \omega$$

$$\omega_k/\omega_{k+1} = \mathcal{S}(\Omega_k) \qquad (\Omega_k = \text{ matrices of rank } k)$$

$$\mathcal{S}(\Omega_k) \simeq \mathrm{Ind}_{\mathrm{GL}(k)\times\mathrm{GL}(m-k)\times\mathrm{GL}(k)\times\mathrm{GL}(n-k)}^{\mathrm{GL}(m)\times\mathrm{GL}(n)}(\mathcal{S}(\mathrm{GL}(k)) \boxtimes 1).$$

Compute

$$\mathrm{Ext}^i_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_k), \pi)$$

By Frobenius reciprocity, Kunneth formula, other basic properties. . .

$$\mathrm{Ext}^i_{\mathrm{GL}(m)}(S(\Omega_k),\pi)^\infty \simeq \sum_{j=1}^{\ell} \mathrm{Ind}^{\mathrm{GL}(n)}_{GL(k)\times GL(n-k)}(\sigma_j \boxtimes 1) \otimes \mathrm{Ext}^i_{\mathrm{GL}(m-k)}(1,\tau_j)$$

$r_{\overline{P}}(\pi) = \sum \sigma_j \boxtimes \tau_j$ implies

Lemma

$\mathrm{Ext}^i_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_k),\pi)$ is a finite length $\mathrm{GL}(n)$-module

$\mathrm{EP}_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_k),\pi)$ is well defined

$\mathrm{EP}_{\mathrm{GL}(m)}(\mathcal{S}(\Omega_k),\pi) = 0$ unless $k = m$.

Theorem

Fix m, and consider the dual pairs $(G = GL(m), GL(n))$ $n \geq 0$.
$\pi \in \widehat{G}$

$$\mathrm{EP}_G(\omega_{m,n}, \pi)^\infty \simeq \begin{cases} 0 & n < m \\ \mathrm{Ind}_P^{GL(n)}(1 \otimes \pi) & n \geq m \end{cases}$$

where $M = GL(n-m) \times GL(m)$

Main Theorem: Type I

Similar idea, using Kudla (and MVW) calculation of the Jacquet module of the oscillator representation

For simplicity: state it for $(\mathrm{Sp}(2m), \mathrm{O}(2n))$ (split orthogonal groups)

$\omega = \omega_{m,n}$ oscillator representation for $(G, G') = (\mathrm{Sp}(2m), \mathrm{O}(2n))$
$t < m \to M(t) = \mathrm{GL}(t) \times \mathrm{Sp}(2m - 2t) \subset \mathrm{Sp}(2m)$

$P(t) = M(t)N(t)$, $\mathrm{Ind}_{P(t)}^{G}()$

$t < n \to M'(t) = GL(t) \times \mathrm{O}(2n - 2t) \subset \mathrm{O}(2m)$

$P'(t) = M'(t)N'(t)$, $\mathrm{Ind}_{P'(t)}^{G'}()$

$\omega_{M(t),M'(t)}$ oscillator representation for dual pair $(M(t), M'(t))$

Main Theorem: Type I

Theorem Fix an irreducible representation π of $M(t)$.

Then

$$\mathrm{EP}_G(\omega_{G,G'}, \mathrm{Ind}_{P(t)}^{G}(\pi))^\infty \simeq \begin{cases} 0 & t > n \\ \mathrm{Ind}_{P'(t)}^{G'}(\mathrm{EP}_{M(t)}(\omega_{M(t),M'(t)}, \pi)^\infty) & t \leq n. \end{cases}$$

$\mathrm{EP}(\omega, \_)^\infty$ commutes with induction