Branching algebras for classical groups

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Survey on some of the works done by Roger Howe and his collaborators (Jackson, Kim, Lee, Tan, Wang, Willenbring) on branching algebras.

Setting:

G : complex classical group
H : certain subgroup of G (mostly symmetric subgroup)

Examples of (G, H): $(\mathrm{GL}_n, \mathrm{O}_n)$, $(\mathrm{Sp}_{2n}, \mathrm{GL}_n)$, $(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n)$

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Branching problem for (G, H)

If V be an irreducible rational G module, what is $V|_H$?

(1) We have

$$V|_H = \bigoplus_U m_{U,V} U$$

where the Us are irreducible H modules.

Determine the branching multiplicities $m(U, V)$.

(2) Describe the H submodules of V.

Use highest weight theory:

Let $B_H = A_H U_H$ be a Borel subgroup of H, and consider

$$V^{U_H} = \{\mathbf{v} : \ g.\mathbf{v} = \mathbf{v} \ \forall g \in U_H\}.$$

This is a module for A_H, and

$$V^{U_H} = \bigoplus_{\lambda} (V^{U_H})_\lambda$$

where

$$(V^{U_H})_\lambda \ = \ \{\mathbf{v} \in V^{U_H} : \ a.\mathbf{v} = \lambda(a)\mathbf{v} \ \forall a \in A_H\}$$

(H highest weight vectors of weight λ)

Then

$$V|_H \simeq \bigoplus_{\lambda} (\dim(V^{U_H})_\lambda) U_\lambda$$

where

U_λ = irreducible H module with highest weight λ.

Branching rule $G \downarrow H$: $$V|_H \simeq \bigoplus_{\lambda} (\dim(V^{U_H})_\lambda) U_\lambda$$

Questions:

1. How to calculate $\dim(V^{U_H})_\lambda$?
2. Can we describe a basis for $(V^{U_H})_\lambda$?

Howe's approach:

(i) Consider a "concrete" algebra $\mathcal{R}_G$ with an G action such that $\mathcal{R}_G$ is decomposed as a multiplicity free sum of irreducible G submodules as

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(iv) Study the branching algebra $\mathcal{A}_{(G,H)}$.

Basic example:

$G = \mathrm{GL}_n \times \mathrm{GL}_n$, $H = \Delta(\mathrm{GL}_n) = \{(g, g) : \ g \in \mathrm{GL}_n\}$.

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Polynomial representations of GL_n are parametrized by Young diagrams with at most n rows (i.e. with depth $\leq n$).

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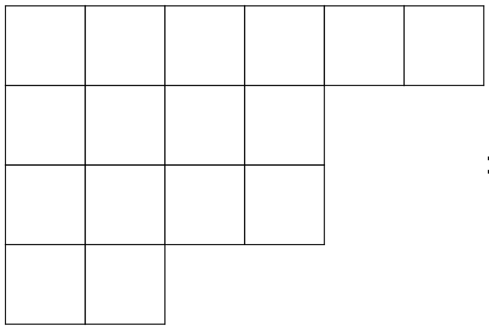
Example of a Young diagram:

$D =$  $=(6,4,4,2)$ or $(6,4,4,2,0)$ etc

Branching problem for $(G, H) = (\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n)$:
For Young diagrams D and E, $\rho_n^D \otimes \rho_n^E$ is an irreducible module for $\mathrm{GL}_n \times \mathrm{GL}_n$.
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So the branching rule in this case is **the Littlewood-Richardson (LR) Rule:**

$$\rho_n^D \otimes \rho_n^E = \bigoplus_F c_{D,E}^F \rho_n^F,$$

where $c_{D,E}^F$ is the number of LR tableaux of shape F/D and content E.

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Then

$$\mathcal{A}_{(G,H)} := \mathcal{R}_G^{U_H} \quad \text{where } \; U_H = U_n = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & * \\ & & \ddots & \\ & 0 & & 1 \end{pmatrix} \in \mathrm{GL}_n \right\}.$$

The construction of $\mathcal{R}_G$:

$\mathrm{GL}_n \times \mathrm{GL}_k$ acts on the algebra $\mathcal{P}(\mathrm{M}_{nk})$ of polynomial functions on $\mathrm{M}_{nk}(\mathbb{C})$:

$$\mathcal{P}(\mathrm{M}_{nk}) \cong \bigoplus_D \rho_n^D \otimes \rho_k^D \qquad (\mathrm{GL}_n, \mathrm{GL}_k) \text{ duality)}$$

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Extracting U_k invariants:

$$\mathcal{P}(\mathrm{M}_{nk})^{U_k} \simeq \bigoplus_D \rho_n^D \otimes \left(\rho_k^D\right)^{U_k} \simeq \bigoplus_D \rho_n^D .$$

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Take another copy:

$$\mathcal{P}(\mathrm{M}_{n\ell})^{U_\ell} \simeq \bigoplus_E \rho_n^E \otimes \left(\rho_\ell^E\right)^{U_\ell} \simeq \bigoplus_E \rho_n^E.$$

Form the tensor product:

$$\mathcal{R}_G := \mathcal{P}(\mathrm{M}_{nk})^{U_k} \otimes \mathcal{P}(\mathrm{M}_{n\ell})^{U_\ell} \simeq \left(\bigoplus_D \rho_n^D \right) \otimes \left(\bigoplus_E \rho_n^E \right) \simeq \bigoplus_{D,E} \rho_n^D \otimes \rho_n^E$$

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Extract the $U_n = \Delta(U_n)$ invariants:

$$\mathcal{A}_{(G,H)} := \mathcal{R}_G^{U_H} = \left(\mathcal{P}(\mathrm{M}_{nk})^{U_k} \otimes \mathcal{P}(\mathrm{M}_{n\ell})^{U_\ell} \right)^{U_n} \simeq \bigoplus_{D,E} \left(\rho_n^D \otimes \rho_n^E \right)^{U_n} .$$

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It can be further decomposed as

$$\mathcal{A}_{(G,H)} \simeq \bigoplus_{D,E} \left\{ \bigoplus_F \left(\rho_n^D \otimes \rho_n^E\right)_F^{U_n} \right\} = \bigoplus_{D,E,F} \mathcal{A}_{(G,H)}^{(D,E,F)}$$

where

$$\mathcal{A}_{(G,H)}^{(D,E,F)} = \left(\rho_n^D \otimes \rho_n^E\right)_F^{U_n} = \text{highest weight vectors of weigth } F \text{ in } \rho_n^D \otimes \rho_n^E$$

$$\dim \mathcal{A}_{(G,H)}^{(D,E,F)} = \text{multiplicity of } \rho_n^F \text{ in } \rho_n^D \otimes \rho_n^E$$

Howe et al. call $\mathcal{A}_{(G,H)}$ a GL_n **tensor product algebra.**

It turns out that $\mathcal{A}_{(G,H)}$ also encodes another branching rule:

$$\begin{aligned}
\mathcal{A}_{(G,H)} &= \mathcal{R}_G^{U_H} = \left(\mathcal{P}(\mathrm{M}_{nk})^{U_k} \otimes \mathcal{P}(\mathrm{M}_{n\ell})^{U_\ell}\right)^{U_n} \simeq \mathcal{P}(\mathrm{M}_{nk} \oplus \mathrm{M}_{n\ell})^{U_n \times U_k \times U_\ell} \\
&\simeq \mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n \times U_k \times U_\ell} \simeq \left(\bigoplus_F \rho_n^F \otimes \rho_{k+\ell}^F\right)^{U_n \times U_k \times U_\ell} \\
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From this, we obtain the reciprocity law:

$\dim \mathcal{A}_{(G,H)}^{(D,E,F)}$ = **multiplicity of** $\rho_k^D \otimes \rho_\ell^E$ **in** ρ_n^F = **multiplicity of** ρ_n^F **in** $\rho_n^D \otimes \rho_n^E$

Problem: Find a basis for $\mathcal{A}_{(G,H)}$.

Since $\mathcal{A}_{(G,H)} = \bigoplus_{D,E,F} \mathcal{A}_{(G,H)}^{(D,E,F)}$, it suffices to find a basis for each subspace $\mathcal{A}_{(G,H)}^{(D,E,F)}$.

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By the Littlewood-Richardson Rule,

$$\begin{aligned}\dim \mathcal{A}_{(G,H)}^{(D,E,F)} &= c_{D,E}^{F} \\ &= \text{number of LR tableaux } T \text{ of shape } F/D \text{ and content } E.\end{aligned}$$

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Plan: LR tableau $T \longrightarrow$ construct a basis vector Δ_T in $\mathcal{A}^{(D,E,F)}_{(G,H)}$

Now

$$\begin{aligned}\mathcal{A}_{(G,H)} &= \left(\mathcal{P}(\mathrm{M}_{nk})^{U_k} \otimes \mathcal{P}(\mathrm{M}_{n\ell})^{U_\ell}\right)^{U_n} \\ &= \mathcal{P}(\mathrm{M}_{n,k} \oplus \mathrm{M}_{n,\ell})^{U_n \times U_k \times U_\ell},\end{aligned}$$

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Write the coordinates of $\mathrm{M}_{n,k} \oplus \mathrm{M}_{n,\ell}$ as

$$\left(\begin{array}{cccc|cccc} x_{11} & x_{12} & \cdots & x_{1k} & y_{11} & y_{12} & \cdots & y_{1\ell} \\ x_{21} & x_{22} & \cdots & x_{2k} & y_{21} & y_{22} & \cdots & y_{2\ell} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nk} & y_{n1} & y_{n2} & \cdots & y_{n\ell} \end{array}\right)$$

Then each Δ_T is a polynomial on these variables.

Associate each skew tableau T with a monomial m_T.

Example: $$T = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \\ \cline{1-2} 2 & & \\ \cline{1-1} \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline x_{11} & x_{11} & y_{11} \\ \hline x_{22} & y_{21} & \\ \cline{1-2} y_{32} & & \\ \cline{1-1} \end{array} \longrightarrow m_T = (x_{11}x_{22}y_{11}y_{32})(x_{11}y_{21})$$

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Introduce a monomial ordering: the graded lexicographic order with

$$x_{11} > x_{21} > \cdots > x_{n1} > x_{12} > \cdots > x_{nk} > y_{11} > y_{21} > \cdots > y_{n\ell}.$$

$$\mathrm{LM}(f) = \text{leading monomial of } f.$$

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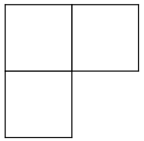
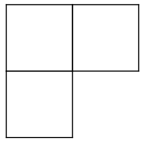
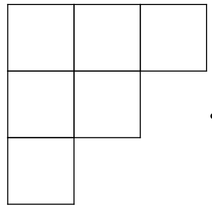
Theorem (Howe-Tan-Willenbring, Advances 2005)

$\mathcal{A}^{(D,E,F)}_{(G,H)}$ *has a basis* $\{\Delta_T\}$ *with the property that for each* T,

$$\mathrm{LM}(\Delta_T) = m_T.$$

Example. Let $D =$  $E =$  $F =$ .

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$$T_1 = \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \quad \Delta_{T_1} = \begin{vmatrix} x_{11} & x_{12} & y_{11} & y_{12} \\ x_{21} & x_{22} & y_{21} & y_{22} \\ x_{31} & x_{32} & y_{31} & y_{32} \\ 0 & 0 & y_{11} & y_{12} \end{vmatrix} \begin{vmatrix} x_{11} & y_{11} \\ x_{21} & y_{21} \end{vmatrix}$$

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$$\mathrm{LM}(\Delta_{T_2}) = (x_{11}x_{22}y_{31})(x_{11}y_{11}y_{22}) = m_{T_2}$$

Let

$$S_{(G,H)} = \{\mathrm{LM}(f) : \ f \in \mathcal{A}_{(G,H)}, \ f \neq 0\} = \{m_T\}.$$

Then $S_{(G,H)}$ is a semigroup because $\mathcal{A}_{(G,H)}$ is an algebra and

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What we can we say about this semigroup $S_{(G,H)}$?

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What we can we say about this semigroup $S_{(G,H)}$?

There is a rational polyhedral cone C in some $\mathbb{R}^N$ such that

$$S_{(G,H)} \simeq C \cap \mathbb{Z}^N.$$

It is finitely generated.

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The polyhedral cone C is called the **Littlewood-Richardson cone** by Igor Pak, and

$c^F_{D,E}$ = number of integral points in a polytope contained in C.

The **initial algebra** $\mathrm{in}(\mathcal{A}_{(G,H)})$ of $\mathcal{A}_{(G,H)}$ is the subalgebra of $\mathcal{P}(\mathrm{M}_{nk} \oplus \mathrm{M}_{nl})$ generated by $S_{(G,H)}$.

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is the **semigroup algebra** on $S_{(G,H)}$, and it is finitely generated.

By a general results of Conca, Herzog, and Valla, we have:

Theorem ([HJLTW]). *The semigroup algebra* $\mathbb{C}[S_{(G,H)}]$ *is a flat deformation of* $\mathcal{A}_{(G,H)}$.

Similar results also hold for the following symmetric pairs (under a stable range condition):

$$(\mathrm{GL}_n, \mathrm{O}_n), \quad (\mathrm{O}_{n+m}, \mathrm{O}_n \times \mathrm{O}_m), \quad (\mathrm{Sp}_{2n}, \mathrm{GL}_n), \quad (\mathrm{GL}_{2n}, \mathrm{Sp}_{2n}),$$

$$(\mathrm{Sp}_{2(n+m)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}), \quad (\mathrm{O}_{2n}, \mathrm{GL}_n)$$

Branching multiplicities in these cases can be deduced from the algebra structure and the LR rule.

m-fold tensor product algebra

This is a branching algebra $\mathcal{A}_{(G,H)}$ which describes the decomposition of m-fold tensor products of GL_n modules:

$$\rho_n^{D_1} \otimes \rho_n^{D_2} \otimes \cdots \otimes \rho_n^{D_m}$$

where

$$G = \mathrm{GL}_n^m, \quad H = \Delta(\mathrm{GL}_n).$$

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A Special case: tensor product of the form

$$\rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)} \simeq \rho_n^D \otimes S^{\alpha_1}(\mathbb{C}^n) \otimes S^{\alpha_2}(\mathbb{C}^n) \otimes \cdots \otimes S^{\alpha_\ell}(\mathbb{C}^n).$$

We call a description of this tensor product **an iterated Pieri rule**.

An algebra which encodes the iterated Pieri rule:

$$\begin{aligned}
\mathcal{P}(\mathrm{M}_{n(k+\ell)}) &= \mathcal{P}(\mathrm{M}_{nk} \oplus \mathbb{C}^n \oplus \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n) \\
&= \mathcal{P}(\mathrm{M}_{nk}) \otimes \mathcal{P}(\mathbb{C}^n) \otimes \mathcal{P}(\mathbb{C}^n) \otimes \cdots \otimes \mathcal{P}(\mathbb{C}^n) \\
&\simeq \left(\bigoplus_{D} \rho_n^D \otimes \rho_k^D \right) \otimes \left(\bigoplus_{\alpha_1} \rho_n^{(\alpha_1)} \right) \otimes \cdots \otimes \left(\bigoplus_{\alpha_\ell} \rho_n^{(\alpha_\ell)} \right) \\
&\simeq \bigoplus_{D,\alpha} \left(\rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)} \right) \otimes \rho_k^D
\end{aligned}$$

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Extract $U_n \times U_k$ invariants:

$$\mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n \times U_k} \simeq \bigoplus_{D,\alpha} \left(\rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)}\right)^{U_n} \otimes \left(\rho_k^D\right)^{U_k}$$

We call $\mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n \times U_k}$ an **iterated Pieri algebra** for GL_n.

The iterated Pieri algebra $\mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n \times U_k}$ also encodes the branching rule for

$$\mathrm{GL}_{k+\ell} \downarrow \mathrm{GL}_k \times \mathrm{GL}_1^\ell = \mathrm{GL}_k \times (\mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1)\,.$$

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Special case: If $k = 1$, then this is branching for

$$\mathrm{GL}_{\ell+1} \downarrow = \mathrm{GL}_1^{\ell+1} = \overbrace{\mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1}^{\ell+1}\,.$$

That is, decompose $\rho^D_{\ell+1}$ into weight spaces, and find a basis of each weight space.

Comparing tensor product algebra with iterated Pieri algebra

GL_n tensor product algebra:

$\mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n \times U_k \times U_\ell}$ describes general tensor products $\rho_n^D \otimes \rho_n^E$.

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We have

$$\mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n \times U_k \times U_\ell} \subseteq \mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n \times U_k}$$

By analyzing how the tensor product algebra sits inside the iterated Pieri algebra, we can give a proof of the Littlewood-Richardson Rule ([Howe-Lee], BAMS 2012).

What is the semigroup S associated with the iterated Pieri algebra $\mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n \times U_k}$?

The elements of S should count the multiplicity in the tensor product $\rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)}$.

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By the Pieri Rule,

$$\rho_p^D \otimes \rho_p^{(\alpha_1)} = \bigoplus_F \rho_p^F \quad \text{(multiplicity free)}$$

where F satisfies the interlacing condition: If $D = (d_1, ..., d_p)$ and $F = (f_1, ..., f_p)$, then

$$f_1 \geq d_1 \geq f_2 \geq d_2 \geq \cdots \geq f_p \geq d_p.$$

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We indicate these inequalities by writing

$$\begin{matrix} & d_1 & & d_2 & & \cdots & & d_p \\ f_1 & & f_2 & & \cdots & & f_p & \end{matrix}$$

By iterating the Pieri Rule,

$$\rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)} = \bigoplus_F m_F \rho_n^F$$

where m_F is the number of "Gelfand-Zeltlin" of the form

$$\lambda = \begin{matrix} & & & \lambda_{10} & & \lambda_{20} & & \cdots & & \lambda_{n0} \\ & & \lambda_{11} & & \lambda_{21} & & \cdots & & \lambda_{n1} & \\ & \iddots & & \iddots & & \cdots & & \iddots & & \\ \lambda_{1\ell} & & \lambda_{2\ell} & & \cdots & & \lambda_{n\ell} & & & \end{matrix}$$

where $D = (\lambda_{10}, \lambda_{20}, \cdots, \lambda_{p0})$ and $F = (\lambda_{1\ell}, \lambda_{2\ell}, \cdots, \lambda_{n\ell})$.

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where $D = (\lambda_{10}, \lambda_{20}, \cdots, \lambda_{p0})$ and $F = (\lambda_{1\ell}, \lambda_{2\ell}, \cdots, \lambda_{n\ell})$.

These patterns can be viewed as **order preserving functions on a poset Γ**

$$\lambda : \Gamma \to \mathbb{Z}^+.$$

The set

$$(\mathbb{Z}^+)^{\Gamma,\geq} = \{f : \Gamma \to \mathbb{Z}^+ | \ f \text{ is order preserving}\}$$

forms a semigroup, and is called a **Hibi cone**. It has a very simple semigroup structure.

(More genearlly, we can replace Γ by a finite poset)

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(More genearlly, we can replace Γ by a finite poset)

Call a subset A of Γ ***increasing*** if

$$a \in A, \ x \in \Gamma, \ x \geq a \Longrightarrow x \in A.$$

Denote by $J^*(\Gamma)$ the collection of all increasing subsets of Γ.

For each $A \in J^*(\Gamma)$, let

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Then clearly $\chi_A \in (\mathbb{Z}^+)^{\Gamma,\geq}$.

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Then clearly $\chi_A \in (\mathbb{Z}^+)^{\Gamma,\geq}$.

Theorem. *The semigroup $(\mathbb{Z}^+)^{\Gamma,\geq}$ is generated by $\{\chi_A : A \in J^*(\Gamma)\}$ and it has relations*

$$\chi_A + \chi_B = \chi_{A\cup B} + \chi_{A\cap B}, \quad A, B \in J^*(\Gamma).$$

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Theorem. *The semigroup $(\mathbb{Z}^+)^{\Gamma,\geq}$ is generated by $\{\chi_A : A \in J^*(\Gamma)\}$ and it has relations*

$$\chi_A + \chi_B = \chi_{A \cup B} + \chi_{A \cap B}, \quad A, B \in J^*(\Gamma).$$

It follows that every $f \in (\mathbb{Z}^+)^{\Gamma,\geq}$ can be expressed as

$$f = \sum_j c_j \chi_{A_j}$$

where $c_j \in \mathbb{N}$ and $A_1 \subset A_2 \subset \cdots \subset A_N = \Gamma$ is a chain in $J^*(\Gamma)$.

In the case when $n = 3$, $k = \ell = 2$, $(\mathbb{Z}^+)^{\Gamma,\geq}$ consists of patterns of the form

$$\lambda = \begin{array}{ccccc} & & \lambda_{10} & \lambda_{20} & 0 \\ & \lambda_{11} & \lambda_{21} & \lambda_{31} & \\ \lambda_{12} & \lambda_{22} & \lambda_{32} & & \end{array}$$

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The generators χ_A of $(\mathbb{Z}^+)^{\Gamma,\geq}$ are:

$$\begin{array}{ccccc} & & 1 & 0 & 0 \\ & 1 & 0 & 0 & \\ 1 & 0 & 0 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & 0 & 0 \\ & 1 & 0 & 0 & \\ 1 & 0 & 0 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & 0 & 0 \\ & 0 & 0 & 0 & \\ 1 & 0 & 0 & & \end{array}$$

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For general n, k, ℓ, each generator χ_A of $(\mathbb{Z}^+)^{\Gamma,\geq}$ corresponds to an element in $\mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n\times U_k}$ of the form

$$\delta_A = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1p} & y_{1s_1} & y_{1s_2} & \cdots & y_{1s_q} \\ x_{21} & x_{22} & \cdots & x_{2p} & y_{2s_1} & y_{2s_2} & \cdots & y_{2s_q} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{(p+q)1} & x_{(p+q)2} & \cdots & x_{(p+q)p} & y_{(p+q)s_1} & y_{(p+q)s_2} & \cdots & y_{(p+q)s_q} \end{vmatrix}.$$

Let Q be the set of all δ_A.

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Let Q be the set of all δ_A.

If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r$, then we call the product

$$\delta_{A_1}\delta_{A_2}\cdots\delta_{A_r}$$

a ***standard monomial*** on Q.

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a ***standard monomial*** on Q.

It turns out that the set of all standard monomials on Q forms a vector space basis for $\mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n \times U_k}$. We say that $\mathcal{P}(\mathrm{M}_{n(k+\ell)})^{U_n \times U_k}$ has a standard monomial theory for Q.

This treatment was given by **Sangjib Kim** in his thesis.

What other branching algebras are associated with Hibi cones?

The double Pieri algebra $\mathcal{L}_{(n,p),(k,q)}$ **for** $\mathrm{GL}_n \times \mathrm{GL}_k$

It describes

$$\left\{\rho_n^D \otimes \left(\otimes_{i=1}^{p} \rho_n^{(\alpha_i)}\right)\right\} \otimes \left\{\rho_k^D \otimes \left(\otimes_{j=1}^{q} \rho_k^{(\alpha_j)}\right)\right\}$$

with $\mathrm{depth}(D) \le k \le n$.

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with $\mathrm{depth}(D) \leq k \leq n$.

The iterated Pieri algebra $\mathcal{A}_{n,k,p}$ **for** O_n **where** $2(k+p) < n$**.**

It describes

$$\sigma_n^D \otimes \left(\otimes_{i=1}^{\ell} \sigma_n^{(\alpha_i)}\right)$$

where σ_n^D is the irreducible representation of O_n labelled by D and $\mathrm{depth}(D) \leq k$.

The iterated Pieri algebra $Q_{n,k,p}$ for Sp_{2n} where $k + p < n$.

It describes

$$\tau_{2n}^{D} \otimes \left(\otimes_{i=1}^{\ell} \tau_{2n}^{(\alpha_i)} \right)$$

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where τ_{2n}^{D} is the irreducible representation of Sp_{2n} labelled by D and $\mathrm{depth}(D) \leq k$.

It turns out that $\mathcal{Q}_{n,k,p} \simeq \mathcal{A}_{2n,k,p}$ for $k + p < n$.

The (more general) iterated Pieri algebra $\mathfrak{A}_{n,k,\ell,p,q}$ for GL_n where $k+p+\ell+q) \leq n$.

It describes

$$\rho_n^{D,E} \otimes \left(\bigotimes_{i=1}^{p} \rho_n^{(\alpha_i)} \right) \otimes \left(\bigotimes_{j=1}^{q} \rho_n^{(\alpha_i)^*} \right)$$

where $\mathrm{depth}(D) \leq k$ and $\mathrm{depth}(E) \leq \ell$.

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where depth$(D) \le k$ and depth$(E) \le \ell$.

It turns out that double Pieri algebras can be regarded as a common structure shared by the iterated Pieri algebras.

Theorem. We have the isomorphism of graded algebras

$$\mathcal{A}_{n,k,p} \simeq \mathcal{L}_{(n,p),(k,p)} \otimes \mathcal{P}(\wedge^2(\mathbb{C}^p)),$$

$$\mathfrak{A}_{n,k,\ell,p,q} \simeq \mathcal{L}_{(n,p),(k,q)} \otimes \mathcal{L}_{(n,q),(\ell,p)} \otimes \mathcal{P}(\mathrm{M}_{pq}).$$

Can the stable range condition be removed?

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Antirow Pieri algebra for GL_n (without stable range condition)

$$\begin{aligned}\mathcal{R}_{n,p,q} &:= \mathcal{P}(\mathrm{M}_{np}) \otimes \left(\bigotimes_{i=1}^{q} \mathcal{P}(\mathbb{C}_i^{n*}) \right) \simeq \left(\bigoplus_{D} \rho_n^D \otimes \rho_p^D \right) \otimes \left(\bigotimes_{i=1}^{q} \rho_n^{(\beta_i)*} \right) \\ &\simeq \bigoplus_{F,\alpha} \left\{ \rho_n^D \otimes \left(\bigotimes_{i=1}^{q} \rho_n^{(\beta_i)*} \right) \right\} \otimes \rho_p^F.\end{aligned}$$

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Extract $\mathrm{GL}_n \times \mathrm{GL}_p$ highest weight vectors:

$$\mathcal{R}_{n,p,q}^{U_n \times U_p} \simeq \bigoplus_{F,\alpha} \left\{ \rho_n^D \otimes \left(\bigotimes_{i=1}^{q} \rho_n^{(\beta_i)*} \right) \right\}^{U_n} \otimes \left(\rho_p^F \right)^{U_p}.$$

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Extract $\mathrm{GL}_n \times \mathrm{GL}_p \times A_q$ highest weight vectors:

$$\mathcal{R}_{n,p,q}^{U_n \times U_p} \simeq \bigoplus_{F,\alpha} \left\{ \rho_n^D \otimes \left(\bigotimes_{i=1}^{q} \rho_n^{(\beta_i)*} \right) \right\}^{U_n} \otimes \left(\rho_p^F \right)^{U_p}.$$

So the algebra $\mathcal{R}_{n,p,q}^{U_n \times U_p}$ describes $\rho_n^D \otimes \left(\bigotimes_{i=1}^{q} \rho_n^{(\beta_i)*} \right)$.

Multiplicities in $\rho_n^D \otimes \left(\bigotimes_{i=1}^{q} \rho_n^{(\beta_i)*} \right)$ are counted by patterns of the form

$$\nu = \begin{array}{cccccccccc} \nu_{10} & & \nu_{20} & & \cdots & & \nu_{n0} & & & \\ & \nu_{11} & & \nu_{21} & & & & \nu_{n1} & & \\ & & \cdot & & \cdot & & & & \cdot & \\ & & & \cdot & & \cdot & & & & \cdot \\ & & & & \nu_{1q} & & \nu_{2q} & & \cdots & \nu_{nq} \end{array}$$

with $D = (\nu_{10}, \nu_{20}, \cdots, \nu_{n0})$.

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Some of the entries ν_{ij} can be negative. The associated semigroup can be identified with a set of order preserving functions $f : \Gamma \to \mathbb{Z}$, and is called a **signed Hibi cone**.

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The structure of the signed Hibi cone and the algebra were determined in Yi Wang's thesis (2013).

Thank you.