Gleason's theorem and unentangled orthonormal bases

Nolan R. Wallach

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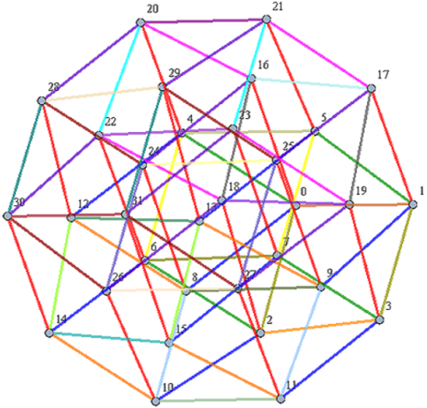
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- Gleason's proof uses a little representation theory, a reduction to 3 real dimensions and geography of the 2 sphere.

A little quantum mechanics

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- To a pure state, v, we form the linear map $v \otimes v^\dagger$. A mixed state is a limit of convex combinations of pure states. Thus mixed state is a positive semidefinite trace class operator of trace 1.
- So Gleason's theorem gives an operational interperatation of mixed states and has been used argue against hidden variables in quantum mechanics.

Two dimensions

- We assume that $\dim \mathcal{H} = 2$. We note that if $v \in \mathcal{H}$ is a unit vector then there is a unique, up to phase, unit vector $\widehat{v}$ orthogonal to it. This yields a map $\mathbb{P}^1 \to \mathbb{P}^1$, $[v] \longmapsto [\widehat{v}]$.

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- Fix such an X and $g : X \to [0,1]$. Then if we define $f(v) = g([v])$ and $f(\widehat{v}) = 1 - g([v])$ then we have defined a frame function and this type of function is the most general one.

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- The phase space for the n the particles is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. If the wave function for the particles is not a product state, that is, not $\phi_1 \otimes \cdots \otimes \phi_n$ with ϕ_i a state in $\mathcal{H}_i$,then the n particles are said to be entangled.

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- If $m > 1, d_i > 1$ then a randomly chosen state will be entangled. Since the dimension of the set of states is $d_1 \cdots d_n - 1$ and the dimension of the set of product states is $d_1 + ... + d_m - n + 1$.Thus if $m > 1$ and all $d_i > 1$ almost all states are entangled.

Unentangled Gleason theorem

- If $\{\phi_{ij}\}_{0 \leq j < \dim \mathcal{H}_i}$ is an orthonormal basis of $\mathcal{H}_i$ then the orthonormal basis $\{\phi_{1i_1} \otimes \cdots \otimes \phi_{ni_n}\}$ is called a product basis. If $\{\varphi_k\}$ is an orthonormal basis and each basis element is a product vector then we will call the basis an unentangled basis.

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- Suppose that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ and we consider $\Sigma = \{\phi_1 \otimes \cdots \otimes \phi_n \mid \|\phi_i\| = 1, \phi_i \in \mathcal{H}_i\}$ and consider the functions $f : \Sigma \to \mathbb{R}_{\ge 0}$ such that for each unentangled basis $\{\varphi_k\}$ we have $\sum f(\varphi_k) = 1$. That is, unentangled frame functions.

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- If $\dim\mathcal{H}_i\geq 3$ for all $i=1,...,n$ then there exists T a mixed state in $\mathcal{H}$ such that $f(\varphi)=\langle T\varphi|\varphi\rangle$ for all product states φ.

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- There is an orthogonal decomposition $V = \bigoplus V_j$ and for each j two orthonormal bases v_{ji} and w_{ji} of V_j and c_j a unit vector in $\mathbb{C}^2$ such that the the basis is

$$\cup_j \left(\{ c_j \otimes v_{ji} \} \right) \cup \left(\{ \widehat{c_j} \otimes w_{ji} \} \right).$$

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- If f is a frame function for $\mathbb{C}^2$ and if g is one for V then $f \otimes g$ is an unentangled one for $\mathbb{C}^2 \otimes V$.

Qubits

- We now concentrate on the case when all of the spaces $\mathcal{H}_i$ are equal to $\mathbb{C}^2$. The case that is most important in quantum information theory. We will denote the Hilbert space $QB_n = \otimes^n \mathbb{C}^2$. From here on I am discussing joint work with Jiri Lebl and Asif Shakeel.

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- A UOB for QB_n is 2^n vectors $z_j = a_1^j \otimes a_2^j \otimes \cdots \otimes a_n^j$, $j = 1, ..., 2^n$ with a_i^j a unit vector in $\mathbb{C}^2$ and for each $j \neq k$ there exists i such that up to phase $a_i^k = \widehat{a_i^j}$. Thus if we think of a UOB projectively $[z_j] = \left[a_1^j\right] \otimes \left[a_2^j\right] \otimes \cdots \otimes \left[a_n^j\right]$ (i.e. using the Segre imbedding of $(\mathbb{P}^1)^n$ into $\mathbb{P}^{2^n-1}$). So the notion of UOB is purely combinatorial.

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- Let S_i be the set $\{[a_i^j] | j = 1, ..., 2^n\}$. We have seen that the involution $S_i \to S_i$ given by $u \longmapsto \hat{u}$ is fixed point free. We can therefore choose $U_i \subset S_i$ a fundamental domain. We write $U_i = \{u_{i1}, ..., u_{ik_i}\}$. We also note that the results above imply that the number of j such that $\left[a_i^j\right] = u$ is the same as the number of j such that $\left[a_i^j\right] = \hat{u}$.

- For each j we put together an n–bit string $b(j)$ with i–th bit 0 if $[a_i^j] = u \in U_i$ or 1 if it is $\hat{u}$ with $u \in U_i$. Set $v_j = e_{b_1} \otimes e_{b_1} \otimes \cdots \otimes e_{b_n}$ with e_0, e_1 the standard orthonormal basis of $\mathbb{C}^2$ and $b = b(j)$.

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- The vectors are orthonormal. Thus there is a permutation, σ, of $1, ..., 2^n$ so that $b(\sigma j)$ is the base two expansion of $j - 1$ padded by 0's on the left. We will say that a UOB is in normal order if σ is the identity. Assume that the UOB is in normal order.

- Set $k = \sum k_i$. We give a coloring of the hypercube graph with k colors corresponding to the UOB. Recall that the n–th hypercube graph has vertices $0, ..., 2^{n-1}$ and edges $[i, j]$ with $i \neq j$ having all binary digits the same except for one $l = l(i, j) = l(j, i)$. We set $m = k_1 + ... + k_{l-1} + p$ if

$$\left[a^i_{l(i,j)}\right] = u, \left[a^j_{l(i,j)}\right] = \hat{u}$$

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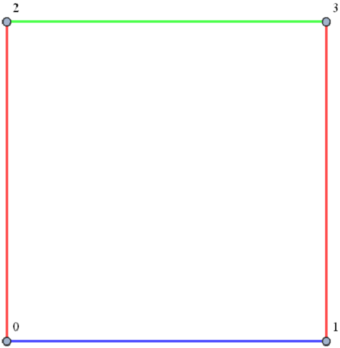
- Set $\alpha^j_i = m$ if the edge eminating from j in the direction i has color m.

$$a \otimes b, a \otimes \hat{b}, \hat{a} \otimes c, \hat{a} \otimes \hat{c}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \end{bmatrix}$$



- $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \end{bmatrix} \Leftrightarrow (u_1, u_2, u_3) \longmapsto$ $\{u_1 \otimes u_2, u_1 \otimes \widehat{u_2}, \widehat{u_1} \otimes u_3, \widehat{u_1} \otimes \widehat{u_3}\}$ a parametrized family of UOB parametrized by $(\mathbb{P}^1)^3$.

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- If we consider the locus of this parameterized subset of the and the one that comes by rotating the square by $\frac{\pi}{2}$ the union of the sets is all UOB in 2 qubits. The intersection of these sets is the locus parameterization that comes from using the red as above but replacing (say) the green by blue.

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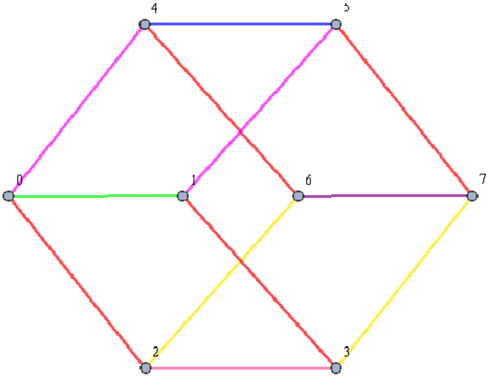
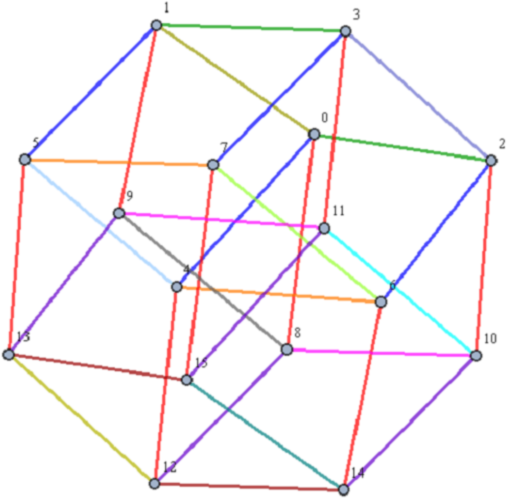
- To get a better idea of the pattern consider Q_3. In this case the maximal number of colors that will come from a UOB is 7 and up to similar rotations to the case above the only coloring with 7 colors coming from a UOB is

cyan, red, green,orange,yellow,brown,blue

$$\begin{bmatrix} 1 & 1 & 5 & 5 & 1 & 1 & 5 & 5 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 4 & 4 & 7 & 7 & 6 & 6 \end{bmatrix}$$

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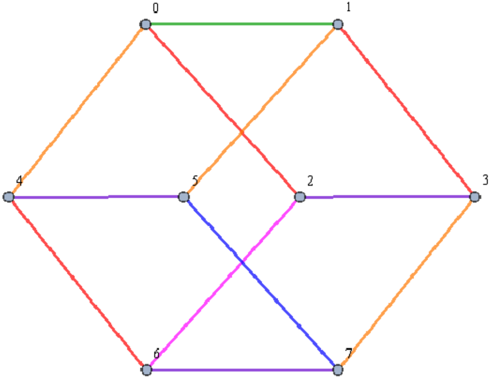
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- We will say that the coloring is useful if it yields a family of UOB. We will say that a useful family is maximal if adding a color makes it useless.

Theorems

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- Note that the directions in the subcube can be different. If we take them to be the same then we have the examples above for $n = 2, 3, 4$.

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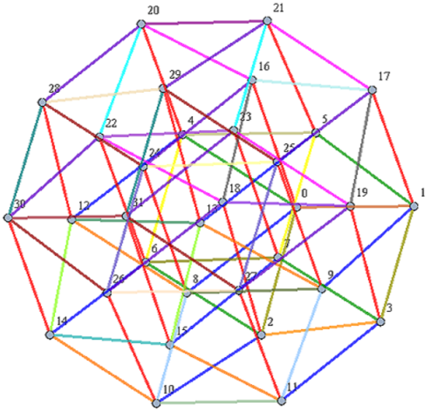
- Any useful coloring has at most $2^n - 1$ colors and this number is attained.
- The useful colorings with $2^n - 1$ colors are constructed recursively as follows: There is one direction, say i, with exactly one color on every edge in that direction coordinates which we call vertical. We then have two $n-1$ cubes Q^0 (i–th coordinate 0) and Q^1 each of these has $2^{n-1} - 1$ with a total of $2^n - 2$ colors (since adding 1 to the total number of colors of the two subcubes is $2^n - 1$. So we see that each has one direction in each subcube with all colors the same and continue.
- Note that the directions in the subcube can be different. If we take them to be the same then we have the examples above for $n = 2, 3, 4$.
- If $n = 3$ then up to permutation of order and permutation of vectors the the two types we gave are all of the maximal useful colorings.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