Relations Between Unitary Representations of Real and p-adic Groups

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(1) G real reductive. Review the Vogan (+ friends) theory of signature characters for Harish-Chandra modules: determining if $\pi \in \widehat{G}$ is unitary.

(2) G (split) p-adic. Review parallel (less well-developed) theory for the graded affine Hecke algebra $\mathbb{H}$ (Barbasch, Ciubotaru, Moy): toward determining if $\pi \in \widehat{G}_{ur}$ is unitary.

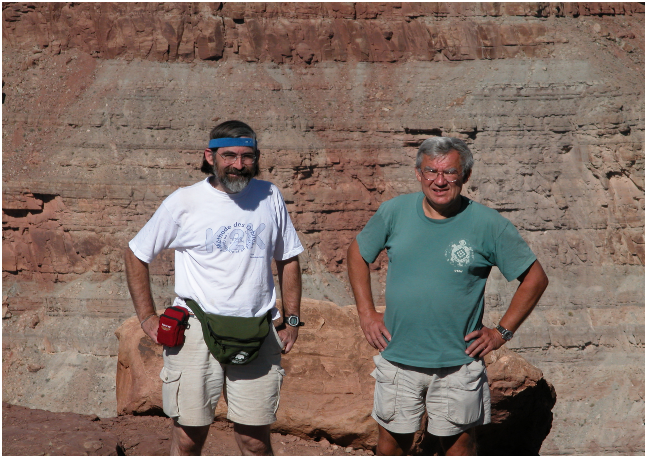
(3) Idea: find relations between spaces of ~~Langlands~~ Adams-Barbasch-Vogan parameters for $G_{\mathbb{R}}$ and Langlands parameters for $G_{\mathbb{Q}_p}$ to develop theory in (2).

(4) A precise theorem in the context of $\mathrm{GL}(n)$.

(ALMOST) NOTHING ABOUT THE ORBIT METHOD

Dubrovnik, 2002

(almost) nothing about the orbit method



Canyonlands, 2004

G complex reductive algebraic group, $G_{\mathbb{R}}$ real form
$K_{\mathbb{R}}$ maximal compact subgroup
$\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, $K = (K_{\mathbb{R}})_{\mathbb{C}} \subset G$
$\sigma :=$ antiholomorphic involution of G defining $G_{\mathbb{R}}$

Definition

A Hermitian form $\langle\ ,\ \rangle$ on a $(\mathfrak{g}, K)$ module X is <u>σ-invariant</u> if

$$\langle Y \cdot u, v\rangle = \langle u, -\sigma(Y) \cdot v\rangle \qquad Y \in \mathfrak{g};$$

$$\langle k \cdot u, v\rangle = \langle u, k^{-1} \cdot v\rangle \qquad k \in K_{\mathbb{R}}.$$

cform

Theorem (Harish Chandra)

Passing to $K_{\mathbb{R}}$-finite vectors induces a bijection from (equivalence classes of) irreducible unitary representations of $G_{\mathbb{R}}$ and (equivalence classes of) irreducible $(\mathfrak{g}, K)$ modules admitting a positive definite σ-invariant form.

Locate unitary dual as a subset of irreducible $(\mathfrak{g}, K)$ modules admitting a positive definite invariant Hermitian form.
Notation:

$$\widehat{G}^u_{\mathbb{R}} \subset \widehat{G}^h_{\mathbb{R}} \subset \widehat{G}_{\mathbb{R}} := \text{irred } (\mathfrak{g}, K) \text{ modules.}$$

Recall that Langlands, Knapp, Zuckerman, Vogan classified $\widehat{G}_{\mathbb{R}}$: to each "parameter" $\gamma \in \mathcal{P}$, there corresponds a standard object $\mathrm{std}(\gamma)$ and canonical irreducible subquotient $\mathrm{irr}(\gamma)$ so that the correspondence

$$\begin{aligned} \mathcal{P} &\longrightarrow \widehat{G}_{\mathbb{R}} \\ \gamma &\longrightarrow \mathrm{irr}(\gamma) \end{aligned}$$

is bijective. Easy to determine if $\mathrm{irr}(\gamma) \in \widehat{G}^h_{\mathbb{R}}$.

A FEW DETAILS OF THE STANDARD MODULES

$P_{\mathbb{R}} = M_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$ cuspidal parabolic subgroup.

σ tempered (limit of discrete series)

e^{ν} character of $A_{\mathbb{R}}$, $\nu \in \mathfrak{a}^*$

typical standard module $\mathrm{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(\sigma \otimes e^{\nu} \otimes 1)$

(e.g. if $\nu = 0$, this is again tempered)

Unitarizability of certain series of representations

By David A. Vogan, Jr.

ON THE EXISTENCE AND IRREDUCIBILITY OF CERTAIN SERIES OF REPRESENTATIONS[1]

BY BERTRAM KOSTANT

Let X be an admissible $(\mathfrak{g}, K)$ module. Set

$$m_X \ : \widehat{K} \longrightarrow \mathbb{Z}$$

be the function that assign $\mu \in \widehat{K}$ the multiplicity of the μ-isotypic space of X:

$$m_X(\mu) = \dim \mathrm{Hom}_K(E_\mu, X) < \infty.$$

When X is irreducible m_X is computable as a consequence of Vogan's proof of the KL conjectures for real groups: In appropriate Grothendieck group, X is a computable sum of standard modules whose m functions are computable.

Let X be a $(\mathfrak{g}, K)$ module with nondegenerate invt form $\langle \ , \ \rangle$. Since $\langle \ , \ \rangle$ is K invariant, it restricts to a nondegenerate form on μ-isotypic space of some signature $(p_X(\mu), q_X(\mu))$. Define the signature character of X to be the function

$$\begin{aligned} \langle \ , \ \rangle_X : \ \widehat{K} &\longrightarrow \mathbb{Z} \oplus \mathbb{Z} \\ \mu &\longrightarrow (p_X(\mu), q_X(\mu)). \end{aligned}$$

Since $m_X = p_X + q_X$, computing $\langle \ , \ \rangle_X$ is a refinement of the branching to K problem.

Notice: If $X \in G^h_{\mathbb{R}}$, then $X \in G^u_{\mathbb{R}}$ if $q \equiv 0$.

Set $\mathbb{W} = \mathbb{Z}[s]/(s^2 = 1)$. Think of

$$\langle\ ,\ \rangle_X : \widehat{K} \longrightarrow \mathbb{W} = \mathbb{Z} \oplus \mathbb{Z}s.$$

Then

$$s \cdot \langle\ ,\ \rangle_X$$

is the "opposite" signature character: if

$$\langle\ ,\ \rangle_X(\mu) = (p, q)$$

then

$$s \cdot \langle\ ,\ \rangle_X(\mu) = (q, p)$$

Theorem

Let X be an irreducible $(\mathfrak{g}, K)$ module with real infinitesimal character and invariant Hermitian form $\langle\ ,\ \rangle$. Then there exist finitely many irreducible tempered modules with real infinitesimal character $Z_1, \cdots, Z_k$ and unique nonzero elements $a_i \in \mathbb{W}$ such that

$$\langle\ ,\ \rangle_X = \sum_i a_i \cdot \langle\ ,\ \rangle_{Z_i}$$

heckecase

If we equip each Z_i with its positive form, then $\langle\ ,\ \rangle_X$ is positive iff all $a_i \in \mathbb{Z}$.

VOGAN'S STRATEGY TO COMPUTE a_i

$$\langle\ ,\ \rangle_X = \sum_i a_i \cdot \langle\ ,\ \rangle_{Z_i}$$

First prove a refinement of the KL conjectures and write

$$\langle\ ,\ \rangle_X = \sum_j b_j \cdot \langle\ ,\ \rangle_{\mathrm{std}(\gamma_j)}$$

as a $\mathbb{W}$ combination of invariant forms of standard modules with the same (real) infinitesimal character.

In terms of computable Jantzen filtrations, Vogan then explicitly described the behavior of

$$\langle\ ,\ \rangle_{\mathrm{std}(\gamma)}$$

as the continuous parameter ν deformed to 0 (where the standard module becomes tempered).

Problem: First step doesn't quite make sense. cform
Overcome in Adams-van Leeuwen-T-Vogan and Yee.

Black box for today: foundational work of Barbasch and Moy (building on the Borel-Casselman equivalence and the Kazhdan-Lusztig classification and further work of Lusztig) gives algebraic reduction of the unramified unitary dual of split reductive p-adic groups to the graded affine Hecke algebra.

Expect that this algebraic setting will handle unitarity questions for much more general kinds of representations (Barbasch-Ciubotaru).

Fix based root system $(\Pi \subset R \subset X, \Pi^\vee \subset R^\vee \subset X^\vee)$.
Set $V = X \otimes_{\mathbb{Z}} \mathbb{C}$, same for $V^\vee$.

$$\mathbb{H} \simeq S(V^\vee) \otimes \mathbb{C}[W] \text{ as a vector space.}$$

Both natural injections of $S(V^\vee)$ and $\mathbb{C}[W]$ in $\mathbb{H}$ are algebra maps. And:

$$\omega t_{s_\alpha} - t_{s_\alpha} s_\alpha(\omega) = (\alpha, \omega), \quad \alpha \in \Pi,\ \omega \in V^\vee.$$

Has natural conjugate linear antiautomorphism τ:

$$\begin{aligned} \tau(t_w) &= t_{w^{-1}}, \quad w \in W, \\ \tau(\omega) &= -\overline{t_{w_0} \omega t_{w_0}}, \quad \omega \in V^\vee. \end{aligned}$$

A Hermitian form $\langle\ ,\ \rangle$ on a $\mathbb{H}$ module X is $\underline{\tau\text{-invariant}}$ if

$$\langle Y \cdot u, v \rangle = \langle u, -\tau(Y) \cdot v \rangle \qquad Y \in \mathbb{H}$$

Notation:

$$\widehat{\mathbb{H}}^u \subset \widehat{\mathbb{H}}^h \subset \widehat{\mathbb{H}} := \text{irred } \mathbb{H} \text{ modules.}$$

Lusztig classified $\widehat{\mathbb{H}}$: once again, to each "parameter" $\gamma \in \mathcal{P}_{\mathbb{H}}$, there corresponds a standard object $\mathrm{std}_{\mathbb{H}}(\gamma)$ and canonical irreducible subquotient $\mathrm{irr}_{\mathbb{H}}(\gamma)$ so that the correspondence

$$\begin{aligned} \mathcal{P} &\longrightarrow \widehat{\mathbb{H}} \\ \gamma &\longrightarrow \mathrm{irr}_{\mathbb{H}}(\gamma) \end{aligned}$$

is bijective. Easy to determine if $\mathrm{irr}_{\mathbb{H}}(\gamma) \in \widehat{\mathbb{H}}^h$.

$$\mathbb{H} = S(V^\vee) \otimes \mathbb{C}[W].$$

Tempered $\mathbb{H}$ modules can be defined according to the Casselman criterion: Z is tempered if the real part of all of the $S(V^\vee)$ generalized weights $\lambda \in V$ are weakly antidominant. The connection with group representations implies they are unitary.

If $\Pi_P \subset \Pi$, can build "parabolic" subalgebra $\mathbb{H}_P$.
Standard modules are of the form

$$\mathbb{H} \otimes_{\mathbb{H}_P} (Z \otimes \mathbb{C}_\nu)$$

where Z is tempered and $\nu \in V$.

SIGNATURE CHARACTER THEOREM FOR $\mathbb{H}$-MODULES

If $\langle\ ,\ \rangle$ is an invariant form on an irreducible $\mathbb{H}$ module, restriction to W isotypic spaces defines a function

$$\langle\ ,\ \rangle_X : \widehat{W} \longrightarrow \mathbb{W} = \mathbb{Z} \oplus s\mathbb{Z}.$$

Theorem

Let X be an irreducible $\mathbb{H}$ module with real central character and invariant Hermitian form $\langle\ ,\ \rangle$. Then there exist finitely many irreducible tempered modules with real central character $Z_1, \cdots, Z_k$ and unique nonzero elements $a_i \in \mathbb{W}$ such that

$$\langle\ ,\ \rangle_X = \sum_i a_i \cdot \langle\ ,\ \rangle_{Z_i}$$

go to real case

No known way to compute the a_i. Main result today: deduce some computations of a_i from the real case. main result

If $G_{\mathbb{R}}$ and $\mathbb{H}$ correspond to split forms arising from the same root data, one might ask if there is a correspondence

$$X \leftrightarrow X' \qquad Z_i \leftrightarrow Z_i'$$

so that the two kinds of formulas have matching coefficients:

$G_{\mathbb{R}}$:

$$\langle\ ,\ \rangle_X = \sum_i a_i \cdot \langle\ ,\ \rangle_{Z_i}\ :\ \widehat{K} \to \mathbb{W}$$

$\mathbb{H}$:

$$\langle\ ,\ \rangle_{X'} = \sum_i a_i \cdot \langle\ ,\ \rangle_{Z_i'}\ :\ \widehat{W} \to \mathbb{W}.$$

Main result: this is exactly true in the case of $\mathrm{GL}(n)$ and suggests generalizations.

FUNCTORS FROM $(\mathfrak{g}, K)$ MODULES TO $\mathbb{H}$-MODULES FOR GL(N)

Set $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$, and let $\mathbb{H}$ correspond to the root system of $\mathfrak{gl}(n)$.

Ciubotaru-T defined exact functors

$$F \;:\; (\mathfrak{g}, K)\text{-mod} \longrightarrow \mathbb{H}\text{-mod}$$

that (when restricted to a nice subcategory) take standard modules to standard modules (or zero) and irreducibles to irreducibles (or zero).

More on this in a moment.

MAIN RESULT

Theorem

In the setting of $\mathrm{GL}(n)$*, suppose* X' *is an irreducible* $\mathbb{H}$ *module with real central character admitting an invariant form. Then there exists an irreducible* $(\mathfrak{g}, K)$ *module* X *admitting an invariant form such that* $F(X) = X'$*. Moreover if*

$$\langle\ ,\ \rangle_X = \sum_i a_i \cdot \langle\ ,\ \rangle_{Z_i}\ :\ \widehat{K} \to \mathbb{W}$$

then

$$\langle\ ,\ \rangle_{F(X)} = \sum_i a_i \cdot \langle\ ,\ \rangle_{F(Z_i)}\ :\ \widehat{W} \to \mathbb{W}.$$

In particular if X *is unitary, then so it* $F(X)$*.*

The proof comes down to ALTV plus a nice relationship between spaces of unramified Langlands parameters for $\mathrm{GL}(n, \mathbb{Q}_p)$ and ABV parameters for $\mathrm{GL}(n, \mathbb{R})$. $+\delta$

$$F \ : \ (\mathfrak{g}, K)\text{-mod} \longrightarrow \mathbb{H}\text{-mod}$$

Following Arakawa-Suzuki, if X is any $U(\mathfrak{g})$ module, there is a natural $\mathbb{H} = S(V^\vee) \otimes \mathbb{C}[W]$ action on

$$X \otimes \overbrace{(\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n)}^{n}$$

The W action is obvious. The action of e_i in $\mathbb{C}^n \simeq V^\vee \hookrightarrow S(V^\vee) \hookrightarrow \mathbb{H}$ involves X and the ith copy of $\mathbb{C}^n$:

$$e_i \mapsto \sum_j [E_j \text{ acting on } X] \otimes [E_j \text{ acting in } ith \text{ component}]$$

where the sum is over any self-dual orthonormal basis $\{E_j\}$ of $\mathfrak{g}$. Commutes with $K_\mathbb{R}$. So get $\mathbb{H}$ action on

$$F(X) := \mathrm{Hom}_{K_\mathbb{R}}\big(1, X \otimes \overbrace{(\mathbb{C}^n \otimes \cdots \mathbb{C}^n)}^{n}\big)$$

$$F \ : \ (\mathfrak{g}, K)\text{-mod} \longrightarrow \mathbb{H}\text{-mod}$$

$$F(X) := \mathrm{Hom}_{K_{\mathbb{R}}}\big(1, X \otimes \overbrace{(\mathbb{C}^n \otimes \cdots \mathbb{C}^n)}^{n}\big)$$

Fairly easy to see (roughly) that F maps standard modules to standard modules (or zero). To see that F maps irreducible to irreducibles (or zero), need some relationship between

geometry of K orbits on G/B (Beilinson-Bernstein)

and

geometry of $G^\vee$ orbits on unramified p-adic
Langlands parameters (Kazhdan-Lusztig, Lusztig)

Not so natural.

$$F \ : \ (\mathfrak{g}, K)\text{-mod} \longrightarrow \mathbb{H}\text{-mod}$$

$$F(X) := \mathrm{Hom}_{K_{\mathbb{R}}}\big(1, X \otimes \overbrace{(\mathbb{C}^n \otimes \cdots \mathbb{C}^n)}^{n}\big)$$

Fairly easy to see (roughly) that F maps standard modules to standard modules (or zero). To see that F maps irreducible to irreducibles (or zero), need some relationship between

geometry of $G^\vee$ orbits on the
Adams-Barbasch-Vogan space.

and

geometry of $G^\vee$ orbits on unramified p-adic
Langlands parameters (Kazhdan-Lusztig, Lusztig)

Ciubotaru-T: Natural relationship exists.

From geometry conclude

decomposition numbers for $\mathbb{H}$ modules

are a subset of

decomposition numbers for $(\mathfrak{g}, K)$ modules.

(This is a real version of a result of Lusztig and Zelvinsky for category $\mathcal{O}$.) Since

$$F \;:\; (\mathfrak{g}, K)\text{-mod} \longrightarrow \mathbb{H}\text{-mod}$$

sends standard modules to the "right" standard modules (or zero), conclude F sends irreducibles to irreducibles (or zero).

IRREDUCIBLE CHARACTERS OF SEMISIMPLE LIE GROUPS IV. CHARACTER-MULTIPLICITY DUALITY

DAVID A. VOGAN, JR.

THEOREM 13.13. *Let G be a real reductive group, and $B = \{cl(\gamma_1), \ldots, cl(\gamma_r)\}$ a block of regular characters of G having nonsingular infinitesimal character. Let $\check{G}$ be a second reductive group, and $\check{B} = \{cl(\check{\gamma}_1), \ldots, cl(\check{\gamma}_r)\}$ a block for $\check{G}$; and assume that the bijection $\gamma_i \to \check{\gamma}_i$ satisfies conditions* (a)–(d) *of Theorem* 11.9. *Then the inverse of the Kazhdan–Lusztig matrix $(P_{\phi\gamma})$ (Lemma* 12.15) *is the matrix*

(a) $(P_{\phi\gamma})^{-1} = ((-1)^{l'(\phi) - l'(\gamma)} P_{\check{\gamma}\check{\phi}}$.

$$G^{\mathbb{C}} = \check{G}^0 AN. \tag{16.6c}$$

This verifies a conjecture of P. Sally that $\check{G}^0 AN$ may be regarded as a complex simple Lie group; I am grateful to him for supplying this example.

In context of $\mathrm{GL}(n)$ and real infinitesimal char, trying to prove

THEOREM

Suppose X' is an irreducible $\mathbb{H}$ module admitting an invariant form. Then there exists an irreducible $(\mathfrak{g}, K)$ module X admitting an invariant form such that $F(X) = X'$. Moreover if

$$\langle\ ,\ \rangle_X = \sum_i a_i \cdot \langle\ ,\ \rangle_{Z_i} \ :\ \widehat{K} \to \mathbb{W}$$

then

$$\langle\ ,\ \rangle_{F(X)} = \sum_i a_i \cdot \langle\ ,\ \rangle_{F(Z_i)} \ :\ \widehat{W} \to \mathbb{W}.$$

In particular if X is unitary, then so it $F(X)$.

using

$$F(X) := \mathrm{Hom}_{K_{\mathbb{R}}}\big(1, X \otimes \overbrace{(\mathbb{C}^n \otimes \cdots \mathbb{C}^n)}^{n}\big)$$

Given $\mathrm{GL}(n,\mathbb{R})$ invariant form $\langle\ ,\ \rangle$ on X, take invariant form $(\ ,\ )$ on $\mathbb{C}^n$ and build invariant form

$$\langle\ ,\ \rangle \otimes (\ ,\ )^{\otimes n} \text{ on } X \otimes (\mathbb{C}^n)^{\otimes n}.$$

Formal consequence of the definition of the $\mathbb{H}$ action shows that this is an $\mathbb{H}$-invariant form and descends to one on

$$F(X) := \mathrm{Hom}_{K_{\mathbb{R}}}\Big(1, X \otimes \overbrace{(\mathbb{C}^n \otimes \cdots \mathbb{C}^n)}^{n}\Big)$$

Jantzen filtrations are preserved by F, etc.

Tiny problem: No $\mathrm{GL}(n,\mathbb{R})$ invariant form exists on $\mathbb{C}^n$.

go to invt form

$\sigma_c :=$
antiholomorphic involution of G defining compact real form

Definition

A Hermitian form $\langle\ ,\ \rangle$ on a $(\mathfrak{g}, K)$ module X is <u>c-invariant</u> if

$$\langle Y \cdot u, v\rangle = \langle u, -\sigma_c(Y) \cdot v\rangle \qquad Y \in \mathfrak{g};$$
$$\langle k \cdot u, v\rangle = \langle u, k^{-1} \cdot v\rangle \qquad k \in K_{\mathbb{R}}.$$

Minor miracle: c-invariant forms exist and can be canonically normalized. This is the starting point of ALTV to complete Vogan's strategy voganstrategy

$\mathbb{H}$ has natural conjugate linear antiautomorphism τ_c:

$$\tau_c(t_w) = t_{w^{-1}}, \quad w \in W,$$
$$\tau_c(\omega) = \overline{\omega}, \quad \omega \in V^\vee.$$

A Hermitian form $\langle \ , \ \rangle$ on a $\mathbb{H}$ module X is <u>c-invariant</u> if

$$\langle Y \cdot u, v \rangle = \langle u, -\tau_c(Y) \cdot v \rangle \qquad Y \in \mathbb{H}$$

Again they exist and are canonical (Barbasch-Ciubotaru).

The proof should be easy...

Given canonical σ_c-invariant form $\langle\ ,\ \rangle^c$ on $(\mathfrak{g}, K)$ module X and $(\ ,\ )^c$ on $\mathbb{C}^n$, build σ_c-invariant form

$$\langle\ ,\ \rangle \otimes (\ ,\ )^{\otimes n} \text{ on } X \otimes (\mathbb{C}^n)^{\otimes n}.$$

Formal consequence of the definition of the $\mathbb{H}$ action shows that this is an τ_c-invariant $\mathbb{H}$ form and descends to one on

$$F(X) := \mathrm{Hom}_{K_{\mathbb{R}}}\big(1, X \otimes \overbrace{(\mathbb{C}^n \otimes \cdots \mathbb{C}^n)}^{n}\big)$$

Easy analysis: Jantzen filtrations are preserved by F, etc.

Now we can follow ALTV to prove a version of the main result for c-invariant forms.

Theorem

In the setting of $\mathrm{GL}(\mathrm{n})$*, suppose* X' *is an irreducible* $\mathbb{H}$ *module with real infinitesimal character. Then there exists an irreducible* $(\mathfrak{g}, K)$ *module* X *such that* $F(X) = X'$*. Moreover the signature characters of the canonical c-invariant forms on* X *and* $F(X)$ *are related as follows: if*

$$\langle\ ,\ \rangle^c_X = \sum_i a_i \cdot \langle\ ,\ \rangle^c_{Z_i}$$

then

$$\langle\ ,\ \rangle^c_{F(X)} = \sum_i a_i \cdot \langle\ ,\ \rangle^c_{F(Z_i)}$$

~~*In particular if* X *is unitary, then so it* $F(X)$~~.

Useless unless we can say something about invariant forms.

FROM c-FORMS TO σ-INVARIANT FORMS

Back to the real case. Recall we had σ_c (defining compact real form) and σ (defining $G_{\mathbb{R}}$). Differ by the Cartan involution for $G_{\mathbb{R}}$

$$\theta = \sigma \circ \sigma_c^{-1}.$$

How can you build a σ-invariant form $\langle\ ,\ \rangle$ on X

$$\langle Y \cdot u, v\rangle = \langle u, -\sigma(Y) \cdot v\rangle$$

from a σ_c-invariant form $\langle\ ,\ \rangle^c$ on X

$$\langle Y \cdot u, v\rangle^c = \langle u, -\sigma_c(Y) \cdot v\rangle^c \ ?$$

Twist by θ: The former exists iff $X \simeq X^\theta$; and if you fix $T : X \to X^\theta$, define

$$\langle u, v\rangle = \langle u, T(v)\rangle^c.$$

Trivial to deduce the σ invariance from the σ_c invariance. So really should be keeping track of this information everywhere.

Always possible to choose "fundamental" δ (automorphism of G preserving K) inner to θ that also preserves a based root datum $(\Pi \subset R \subset X, \Pi^\vee \subset R^\vee \subset X^\vee)$. Consider representations of the extended group $G^\delta = G \rtimes \{1, \delta\}$, that is $(\mathfrak{g}, K^\delta)$ modules.

For $(\mathfrak{g}, K^\delta)$ module X, can translate canonical c-invariant form into $G_\mathbb{R}$ invariant form.

On the Hecke algebra side, δ also gives an automorphism of $\mathbb{H}$. Consider modules for "extended" algebra $\mathbb{H}^\delta = \mathbb{H} \rtimes \{1, \delta\}$.

Same formalism shows that action of δ relates c-invariant forms on $\mathbb{H}^\delta$ modules to honest invariant forms.

BACK TO GL(N) TO FINISH THE PROOF

Redo the functors for groups and algebras extended by fundamental automorphism δ:

$$F \ : \ (\mathfrak{g}, K^\delta)\text{-mod} \longrightarrow \mathbb{H}^\delta\text{-mod}$$

The new results of Lusztig-Vogan imply

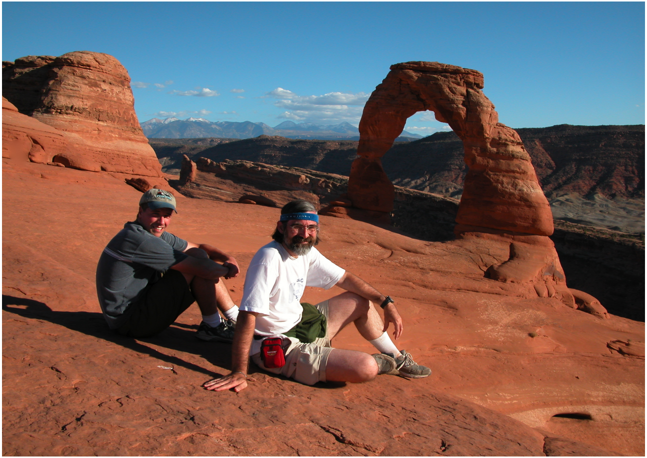
decomposition numbers for $\mathbb{H}^\delta$ modules

are related to a subset of

decomposition numbers for $(\mathfrak{g}, K^\delta)$ modules.

So the functors once again have good properties, preserve Jantzen filtrations, etc.

The rest is just following your nose....



Happy Birthday!