Transfer results for real groups

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G : **a connected reductive algebraic group defined over** $\mathbb{R}$

- study some aspects of invariant harmonic analysis on $G(\mathbb{R})$ involved in endoscopic transfer

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- study some aspects of invariant harmonic analysis on $G(\mathbb{R})$ involved in endoscopic transfer

- transfer: first for orbital integrals [geometric side], then look for interpretation of the dual transfer in terms of traces [spectral side]

part of broader theme involving stable conjugacy, packets of representations, stabilization of the Arthur-Selberg trace formula, ...

concerned here with explicit structure, formulas useful in applications

Setting has more than G alone

- start with quasi-split data: G^* quasi-split group over $\mathbb{R}$

  fix [harmlessly] an $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X_\alpha\})$
  ... $\Gamma = Gal(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ preserves each component

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- then have connected complex dual group $G^\vee$ and
  dual splitting $spl^\vee = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$
  that is preserved by the real Weil group $W_\mathbb{R}$

  here $W_\mathbb{R}$ acts on $G^\vee$ and $spl^\vee$ through $W_\mathbb{R} \to \Gamma$
  then L-group ${}^LG := G^\vee \rtimes W_\mathbb{R}$

  [from $W_\mathbb{R}$ action: toral chars with special symms for geom side of transfer, shifts in inf char for spectral side, etc]

G as inner form of a quasi-split G^*

- consider pair (G, η), where isom $\eta : G \longrightarrow G^*$ is inner twist
  i.e. the automorphism $\eta\ \sigma(\eta)^{-1}$ of G^* is inner

  inner class of (G, η) consists of (G, η') with $\eta'\eta^{-1}$ inner
  [inner class is what matters in constructions here]

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- $t(G) :=$ set of [stable] conjugacy classes of maximal tori defined over $\mathbb{R}$ in G

  $t(G)$ as lattice: *class*$(T) \preccurlyeq$ *class*$(T') \Leftrightarrow$ maximal $\mathbb{R}$-split subtorus S_T of T is $G(\mathbb{R})$-conjugate to a subtorus of $S_{T'}$

  Proposition: η embeds $t(G)$ in $t(G^*)$ as an initial segment

Endoscopic group H_1 comes from certain dual data [SED]

- semisimple element s in $G^\vee$
  $H^\vee := Cent(s, G^\vee)^0$
  subgroup $\mathcal{H}$ of ${}^L G$ that is split extension of $W_\mathbb{R}$ by $H^\vee$ ...

  extract L-action, L-group ${}^L H$ and thus dual quasi-split group H over $\mathbb{R}$, pass to z-extension H_1
  $[1 \to Z_1 \to H_1 \to H \to 1$, with Z_1 central induced torus$]$

SED $\mathfrak{e}_z$: $(s, \mathcal{H}, H)$ and (H_1, ξ_1), where

$$\mathcal{H} \xrightarrow{\xi_1} {}^L H_1, \qquad \mathcal{H} \xrightarrow{incl} {}^L G$$

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- transfer involves $H_1(\mathbb{R})$ and $G(\mathbb{R})$, for each inner form (G, η) of G^* [funct. says ...]

Geometric comparisons for $H_1(\mathbb{R})$ and $G(\mathbb{R})$

- $$t(H_1) \searrow t(H) \longrightarrow t(G^*) \swarrow t(G)$$

  via maps on maximal tori: (i) z-extension $H_1 \rightarrow H$
  (ii) admissible homs $T_H \rightarrow T_{G^*}$ [defn SED, Steinberg thm],
  (iii) inner twist $\eta : G \rightarrow G^*$

  or as Γ-equivariant maps on semisimple conjugacy classes in complex points of the groups. Strongly reg class in G or G^*: centralizer of element is torus. Strongly G-reg class in H_1 or H: image of class is strongly regular in G^*

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- the very regular set in $H_1(\mathbb{R}) \times G(\mathbb{R})$: pairs (γ_1, δ) with δ strongly regular in $G(\mathbb{R})$, γ_1 strongly G-regular in $H_1(\mathbb{R})$

Related pairs of points

- very regular pair (γ_1, δ) is related if there is δ^* in $G^*(\mathbb{R})$ for which $\gamma_1 \longrightarrow \gamma \longrightarrow \delta^* \longleftarrow \delta$
  [alt: γ_1 is an image/norm of δ]

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- generalize to all semisimple pairs (γ_1, δ), then related (γ_1, δ) is *equisingular* if $Cent(\gamma, H)^0$ is an inner form of $Cent(\delta, G)^0$ ...

  outside equisingular set: e.g. related pairs (u_1, u), with u_1, u regular unipotent in $H_1(\mathbb{R}), G(\mathbb{R})$ respectively if G is quasi-split, or more generally (γ_1, δ) regular ...

  but need transfer factors $\Delta(\gamma_1, \delta)$ only on very regular set to fully define transfer of test functions on geometric side, then others via limit thms etc, all local fields of char zero ...

Related pairs of representations

- arrange pairs (π_1, π) on spectral side via Langlands/Arthur parameters

  L: consider continuous homs $w \mapsto \varphi(w) = \varphi_0(w) \times w$ of $W_{\mathbb{R}}$ into ${}^L G = G^\vee \rtimes W_{\mathbb{R}}$,
  require image of φ_0 lie in semisimple set, bounded mod center; $G^\vee$ acts by conjugation on such homs,
  essentially tempered parameter is [relevant] $G^\vee$-conj. class

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- related pairs: back to SED, require (π_1, π) have related parameters (φ_1, φ) $[\varphi_1(W_{\mathbb{R}}) \subset \xi_1(\mathcal{H}),\ \varphi \sim \xi_1^{-1} \circ \varphi_1]$

  very regular related pair (π_1, π): also require $Cent(\varphi(\mathbb{C}^\times), G^\vee)$ abelian, then also $Cent(\varphi_1(\mathbb{C}^\times), H^\vee)$ abelian ... regular infinitesimal chars

Extending ...

- transfer factors first for very regular related pairs (π_1, π) of ess. tempered representations

  extend to all ess. tempered; then recapture (for given pair test functions) geom side from ess. temp spectral side

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- A: consider $G^\vee$-conj. classes of continuous homs $\psi = (\varphi, \rho) : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \to {}^L G$, with φ as before ...

  $M^\vee := Cent(\varphi(\mathbb{C}^\times), G^\vee)$ is Levi in $G^\vee$, contains image of ρ
  $\mathcal{M} :=$ subgroup of ${}^L G$ generated by $M^\vee$ and image of ψ

Some pairs of Arthur parameters

- call ψ u-regular if image of ρ contains reg unip elt of $M^\vee$; ess tempered $\psi = (\varphi, triv)$ is u-regular $\Leftrightarrow$ φ regular

  consider pairs (ψ_1, ψ) related (as before) and with ψ u-regular, then ψ_1 also u-regular; components φ_1, φ are equi-singular in approp sense

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- structure of $\mathcal{M}$: extract L-action and L-group ${}^L M$
  $M^\vee$ is Levi $\Rightarrow$ *natural* isomorphisms ${}^L M \to \mathcal{M}$

  $M^* :=$ quasi-split group over $\mathbb{R}$ dual to ${}^L M$
  M^* *shares* elliptic maximal torus with G $\Leftrightarrow$
  there is elt of $\mathcal{M}$ acting as -1 on all roots of $\mathit{spl}^\vee$

Cuspidal-elliptic setting

- elliptic parameter ψ [Arthur]: identity component of centralizer in $G^\vee$ of $Image(\psi)$ is central in $G^\vee$
  [ess tempered case: only parameters for discrete series]

  elliptic u-regular ψ points to packets constructed by Adams-Johnson, plus either discrete series or limit of discrete series packets, same inf char

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- G cuspidal: has elliptic maximal torus
  SED $\mathfrak{e}_z$ elliptic: identity comp of Γ-invariants in the center of $H^\vee$ is central in $G^\vee$... call this *cuspidal-elliptic setting*

  Proposition: in cusp-ell setting have elliptic u-regular related pairs (ψ_1, ψ), and only in this setting

Test functions, measures

- on $G(\mathbb{R})$: Harish-Chandra Schwartz functions, then $C_c^\infty(G(\mathbb{R}))$, also subspaces of K-finite ...

  on $H_1(\mathbb{R})$: corresponding types of test functions but modulo $Z_1(\mathbb{R}) = Ker(H_1(\mathbb{R}) \to H(\mathbb{R}))$,
  SED $\mathfrak{e}_z$ determines character ϖ_1 on $Z_1(\mathbb{R})$: require translation action of $Z_1(\mathbb{R})$ on test functions is via $(\varpi_1)^{-1}$

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- use test measures fdg and $f_1 dh_1$ to remove dependence of transfer on choice of Haar measures dg, dh_1

  compatible haar measures on tori associated to very regular related pair of points (γ_1, δ) ... via $T_1 \rightarrow T_H \rightarrow T$

Transfer factors

- $(\gamma_1, \delta), (\gamma_1', \delta')$ very regular related pairs of points, define certain canonical relative factor $\Delta(\gamma_1, \delta; \gamma_1', \delta')$ as product of three terms: $\Delta = \Delta_I . \Delta_{II} . \Delta_{III}$ [all depend only on stable conj cls γ_1, γ_1', and conj cls δ, δ']

  terms $\Delta_I, .., \Delta_{III}$ each have two additional dependences that cancel in product; only Δ_{III} genuinely relative, measures position in stable class, other terms make this canonical

  $(\pi_1, \pi), (\pi_1', \pi')$ very regular related ess temp pairs, define canonical relative factor $\Delta(\pi_1, \pi; \pi_1', \pi')$ as product of three ... [all depend only on packets of π_1, π_1', and on π, π']

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- geom, spec terms have same structure and dependences $\Rightarrow$ canonical $\Delta(\gamma_1, \delta; \pi_1, \pi)$ also

Compatibility

- absolute factors $\Delta(\gamma_1, \delta)$ and $\Delta(\pi_1, \pi)$:
  require for all above pairs

$$\begin{aligned} \Delta(\gamma_1, \delta)/\Delta(\gamma_1', \delta') &= \Delta(\gamma_1, \delta; \gamma_1', \delta') \\ \Delta(\pi_1, \pi)/\Delta(\pi_1', \pi') &= \Delta(\pi_1, \pi; \pi_1', \pi') \end{aligned}$$

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- particular normalizations not needed in main theorem
  used later for structure results, precise inversion results ...

  data for statement of main theorem: quasisplit group, SED, inner form, compatible factors

Theorem

- *For each test measure fdg on $G(\mathbb{R})$ there exists a test measure $f_1 dh_1$ on $H_1(\mathbb{R})$ such that*

$$SO(\gamma_1, f_1 dh_1) = \sum_{\{\delta\}} \Delta(\gamma_1, \delta)\, O(\delta, fdg) \qquad (1)$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$.

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- *Then also*

$$St\text{-}Trace\; \pi_1(f_1 dh_1) = \sum_{\{\pi\}} \Delta(\pi_1, \pi)\; Trace\; \pi(fdg) \tag{2}$$

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- *Conversely if fdg and $f_1 dh_1$ satisfy (2) then they satisfy (1).*

More on (1) : $SO(\gamma_1, f_1 dh_1) = \sum_{\{\delta\}} \Delta(\gamma_1, \delta)\ O(\delta, fdg)$

- $\Delta(\gamma_1, \delta) := 0$ if very regular pair (γ_1, δ) is not related, then sum on right is over str reg conjugacy classes $\{\delta\}$

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  where $T_\delta = Cent(\delta, G)$

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  where the sum is over conjugacy classes $\{\gamma_1'\}$ in the stable conjugacy class of γ_1, compatible measure $dt_{\gamma_1'}$ on $T_{\gamma_1'}$

More on (2): *St-Trace* $\pi_1(f_1 dh_1) = \sum_\pi \Delta(\pi_1, \pi)$ *Trace* $\pi(fdg)$

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- $\Delta(\pi_1, \pi)$ has been extended to all ess tempered pairs (π_1, π). Also $\Delta(\pi_1, \pi) := 0$ if pair (π_1, π) is not related, then sum on right is over all ess tempered π

Next steps

- geom side: begin extensions as already mentioned, ... new factors involve more general invariants, will use "same type of structure" on spectral side

  spec side: first (1) with particular normalizations ... in general, $\Delta(\pi_1, \pi)$ is a fourth root of unity, up to constant, on all tempered related pairs

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- Whittaker data for quasi-split G: $G(\mathbb{R})$-conjugacy class of pairs (B, λ) : $B =$ Borel subgroup defined over $\mathbb{R}$, $\lambda =$ character on real points of unipotent radical of B [harmless: $G = G^*$, (B, λ) from *spl** via add char $\mathbb{R}^\times$]

Normalization of transfer factors

- define compatible absolute factors $\Delta_{Wh}(\gamma_1, \delta)$, $\Delta_{Wh}(\pi_1, \pi)$
[quasi-split case: have compatible absolute factors Δ_0 depending on *spl**; multiply each by certain ε-factor]

Proposition: For all essentially tempered related pairs (π_1, π), we have

$$\Delta_{Wh}(\pi_1, \pi) = \pm 1.$$

note: on geom side, for very regular (γ_1, δ) near $(1, 1)$, we have the shape $\Delta_{Wh}(\gamma_1, \delta) = [\mathit{sign}].[\varepsilon].[\mathit{shift\text{-}char}]$

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- normalization extends to inner forms (G, η) such that $\eta\ \sigma(\eta)^{-1} = Int(u(\sigma))$, where $u(\sigma)$ is cocycle in G^*_{sc}.

Structure on essentially tempered packets ...

- begin with cuspidal-elliptic setting, elliptic parameter φ

  $S_\varphi := Cent(Image\ \varphi, G^\vee)$ for Langl φ (sim for Arthur ψ)

  Example: G^* simply-connected semisimple, so $G^\vee$ adjoint, recall splitting $spl^\vee = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $G^\vee$. Then: since φ elliptic we can arrange $S_\varphi =$ elts in $\mathcal{T}$ of order $\leqslant 2$

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- for each (G, η) with Whitt norm, consider π in packet for φ, then we will identify π with a character on S_φ ... get certain *extended* packet for G^* as dual of quotient of S_φ [ess temp] **In example**: extended packet is exactly dual of S_φ

  will use particular case of construction from twisted setting

Fundamental splittings

- recall: $\mathbb{R}$-splitting $spl^* = (B^*, T^*, \{X_\alpha\})$ for G^* with dual $spl^\vee$ for $G^\vee$

for any G and T fundamental maximal torus in G :
pair (B, T) fundamental if $-\sigma$ preserves roots T in B;
there is a single stable conj. class of such pairs

fund splitting: extend fund pair (B, T) to splitting
$spl = (B, T, \{X_\alpha\})$ where simple triples $\{X_\alpha, H_\alpha, X_{-\alpha}\}$
are chosen ($H_\alpha =$ coroot) and $\sigma X_\alpha = X_{\sigma\alpha}$ if $-\sigma\alpha \neq \alpha$,
$\sigma X_\alpha = \varepsilon_\alpha X_{-\alpha}$, where $\varepsilon_\alpha = \pm 1$ otherwise

any two extensions of (B, T) are conjugate under $T_{sc}(\mathbb{R})$
Whittaker data determines fund splitting spl_{Wh} for G^*

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- back to cusp-ell setting: attach fundamental spl_π [or pair] to elliptic π via Harish-Chandra data

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- now form extended group [K-group] of quasi-split type:
  $\mathbf{G} := G_0 \sqcup G_1 \sqcup G_2 \sqcup \ldots \sqcup G_n$
  [harmless] take $(G_0, \eta_0, u_0(\sigma)) = (G^*, id, id)$

  general components of $\mathbf{G}$: take cocycles $u_j(\sigma)$ in T_{sc} representing the fibers of $H^1(\Gamma, G^*_{sc}) \to H^1(\Gamma, G^*)$ and then (G_j, η_j) with $\eta_j \sigma(\eta_j)^{-1} = Int\ u_j(\sigma)$ $[spl_{Wh} = (B, T \ldots]$

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- spl_π is determined uniquely up to $G(\mathbb{R})$-conjugacy
- now form extended group [K-group] of quasi-split type:
  $\mathbf{G} := G_0 \sqcup G_1 \sqcup G_2 \sqcup \ldots \sqcup G_n$
  [harmless] take $(G_0, \eta_0, u_0(\sigma)) = (G^*, id, id)$

  general components of $\mathbf{G}$: take cocycles $u_j(\sigma)$ in T_{sc} representing the fibers of $H^1(\Gamma, G^*_{sc}) \to H^1(\Gamma, G^*)$ and then (G_j, η_j) with $\eta_j \sigma(\eta_j)^{-1} = Int\ u_j(\sigma)$ [$spl_{Wh} = (B, T \ldots$]
- form $\mathbf{\Pi} := \Pi_0 \sqcup \Pi_1 \sqcup \Pi_2 \sqcup \ldots \sqcup \Pi_n$
  as extended packet for [ess temp] φ
  $\mathbf{\Pi}$ correct size ... G^* **scss:** $|\mathbf{\Pi}| = |H^1(\Gamma, T)|$

Invariants ...

- **Example**: **write theorem for the case G^* scss**
  consider component G_j and rep $\pi = \pi_j$ of $G_j(\mathbb{R})$ in Π_j

  there is unique $\eta_\pi = Int(x_\pi) \circ \eta_j$, where $x_\pi \in G^*_{sc} = G^*$,
  that transports spl_{π_j} to spl_{Wh} ... then
  $v_\pi(\sigma) := x_\pi u_j(\sigma)\sigma(x_\pi)^{-1}$ has $\eta_\pi\sigma(\eta_\pi)^{-1} = Int\ v_\pi(\sigma)$

  $inv(\pi) :=$ class of cocycle $v_\pi(\sigma)$ in $H^1(\Gamma, T)$
  $\pi \mapsto inv(\pi)$: well-defined, bijective $\mathbf{\Pi} \to H^1(\Gamma, T)$

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- $spl^\vee = (spl^*)^\vee$ and $spl^* \to spl_{Wh}$ provide $\mathcal{T} \to T^\vee$
  under which S_φ isom to $(T^\vee)^\Gamma$; write s_T for image of s

  recall Tate-Nakayama duality provides perfect pairing

$$\langle -, - \rangle_{tn} : H^1(\Gamma, T) \times (T^\vee)^\Gamma \to \{\pm 1\}$$

Apply to transfer

- now have perfect pairing $\mathbf{\Pi} \times S_\varphi \to \{\pm 1\}$ given by

$$(\pi, s) \mapsto \langle \pi, s \rangle := \langle inv(\pi), s_T \rangle_{tn}$$

note: $s \mapsto \langle \pi, s \rangle$ trivial char when π is unique generic in $\mathbf{\Pi}$ for given Whittaker data

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- for $s \in S_\varphi$: construct elliptic SED $\mathfrak{e}_s = (s, \dots)$, with
  $\mathcal{H}_s :=$ subgroup of ${}^L G$ generated by $Cent(s, G^\vee)^0$
  and the image of φ,
  along with a preferred related pair (φ_s, φ)

  use Whitt. norm for transfer from attached endoscopic group $H_1^{(s)}$ to $\mathbf{G}$

- **Theorem (strong basepoint property):**

$$\Delta_{Wh}(\pi_s, \pi) = \langle \pi, s \rangle$$

Corollary:

$$Trace\ \pi(fdg) = \left|S_\varphi\right|^{-1} \sum_{s \in S_\varphi} \ \langle \pi, s \rangle \ \ St\text{-}Trace\ \pi_s(f_1^{(s)} dh_1^{(s)})$$

thm for any G of quasi-split type [drop $G^* = G^*_{sc}$] **and** φ **ess bded parameter :** replace S_φ by quotient, extend defn pairing $\langle \pi, * \rangle$ ... need uniform decomp of unit princ series

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- **general inner forms**: can arrange in extended groups but lack natural basepoint; pick bp (G, η) and use Kaletha's norm of transfer factors on this group [rigidify $\{\eta\}$, refine EDS $\mathfrak{e}_z$]

  then can norm all factors for extended group and get variant of quasi-split structure; ... but need Kaletha's cohom theory to identify explicitly all constants in transfer

Data attached to u-regular parameter

- back to cusp-ell setting, method for spectral factors extends: now take ψ elliptic u-regular Arthur param

  transport explicit data for ψ to elliptic T in G^*
  [as for s, use $spl^\vee$ dual $spl^*, spl^* \longrightarrow spl_{Wh} = (B, T, \{X_\alpha\})$]

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- explicit data: have $\psi = (\varphi, \rho)$ and $\xi_M : {}^LM \to \mathcal{M}$
  there is an (almost) canonical form:
  $\varphi(z) = z^\mu \overline{z}^{\sigma_M \mu} \times z$ for $z \in \mathbb{C}^\times$ and
  $\varphi(w_\sigma) = e^{2\pi i \lambda} \xi_M(w_\sigma)$, where $w_\sigma \to \sigma$ and $w_\sigma^2 = -1$

  $\mu, \lambda \in X_*(\mathcal{T}) \otimes \mathbb{C}$ have several special properties ...
  these determine a character on $M^*(\mathbb{R})$ and inner forms,
  also particular (s-)elliptic parameter [$M^* \longrightarrow$]

Attached packets

- M^* as subgroup of G^* generated by T and coroots for $M^\vee$ as roots of T in G^* is quasi-split Levi group [in *Cart*-stable parabolic of G^*, $Cart = Int(t_0)$, $t_0 \in T(\mathbb{R})$]

  Arthur packet for inner form (G, η): use any η' inner to η with $(\eta')^{-1} : T \to G$ defined over $\mathbb{R}$ to transport data for ψ to certain *character data* for G, gather reps so defined

  character data: for irred ess unitary repn cohom induced from character on $M'(\mathbb{R})$, where M' is twist of M^* by η' ... this is packet defined by Adams-Johnson

  also get discrete series or limit packet, same inf char

Transfer for these packets ...

- now $S_\psi = \Gamma$-invariants in $Center(M^\vee) \subseteq \Gamma$-invariants in T
  example: extended group $\mathbf{G}$ of quasi-split type, scss

  attach M_π, spl_π, $q_\pi = q(M_\pi)$ to $\pi \in \mathbf{\Pi}$
  $inv(\pi)$ well-defined up to cocycles generated by roots of $M^\vee$ as coroots for T, so that $\langle \pi, s \rangle := \langle inv(\pi), s \rangle_{tn}$ well-def

  extend relative spectral factors, recover identities from Adams-Johnson, Arthur, Kottwitz results ...

KS setup, briefly

- same approach to examine twisted setting

  quasi-split data now includes $\mathbb{R}$-automorphism θ^* of G^* that preserves $\mathbb{R}$-splitting spl^*, finite order
  [also dual datum for tw char ϖ on real pts any inner form]

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- inner form (G, η, θ) : includes $\mathbb{R}$-automorphism θ of G such that η transports θ to θ^* up to inner automorphism

  inner class of (G, η, θ) : (G, η', θ'), where η' inner form of η, θ coincides with θ' up to inner autom by element of $G(\mathbb{R})$

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- transfer: stable analysis on endo gp $H_1(\mathbb{R})$ related to θ-twisted invariant analysis on $G(\mathbb{R})$ [(θ, ϖ)-twisted]

Cuspidal-elliptic case, geom side

- point correspondences now via $T \rightarrow (T)_{\theta^*} \longleftrightarrow T_H \longleftarrow T_1$
  norm for (G^*, θ^*) is canonical; not for general (G, η, θ) but ...

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- geom side: call $\delta \in G(\mathbb{R})$ θ-elliptic if $Int(\delta) \circ \theta$ preserves a pair (B, T), where T is elliptic

  for ell very reg contribution: call very regular pair (γ_1, δ) elliptic if γ_1 is elliptic

  Proposition: there exists an elliptic related very regular pair if and only if $G(\mathbb{R})$ contains a θ-elliptic elt ... then "full" ell csp

spectral side

- contribution from ess tempered elliptic (ds) packets
  quasi–split data (G^*, θ^*): pick θ^*-stable Whitt. data

  θ^* has dual $\theta^\vee$, extend to ${}^L\theta$ which acts on parameters,
  interested only those φ (conj class) preserved by ${}^L\theta$,
  i.e. $S_\varphi^{tw} = \{s \in G^\vee : {}^L\theta \circ \varphi = Int(s) \circ \varphi\}$ is nonempty

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  $\pi \to \pi \circ \theta$ but twist-packet may be empty
- **Proposition**: there exists nonempty ds twist-packet if and
  only if $G(\mathbb{R})$ has a θ-elliptic elt ... and then all ds twist-pkts
  nonempty

- **Proof** of second proposition via Harish-Chandra theory for discrete series. For first proposition use following:

  Lemma: $\exists$ θ-elliptic elt $\Leftrightarrow$ there is (G, η', θ') in the inner class of (G, η, θ) such that θ' preserves a fundamental splitting and η' transports θ' to θ^*

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- **Application:** for elliptic analysis, may assume θ preserves a fundamental splitting that is transported by η to spl_{Wh} [have fund Whittaker splitting spl_{Wh} preserved by θ^*]

  then uniquely defined norm, also proceed as before for spectral factors [Mezo, Waldspurger for spec transf exists], compatibility results ... ■