UPPER SEMICONTINUITY OF KLV POLYNOMIALS FOR CERTAIN BLOCKS OF HARISH-CHANDRA MODULES

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G = complex connected reductive group

B = Borel subgroup

G/B = flag variety

K = spherical subgroup of G (so that $K\backslash G/B$ is finite)

$\bar{\mathcal{O}}$ = closure of K-orbit in G/B

Mainly interested in the case of symmetric K (the fixed points of an involution), but begin with the more familiar case $K = B$.

$\bar{\mathcal{O}}$ = Schubert variety

Here $\bar{\mathcal{O}} = \bar{\mathcal{O}_w}, w \in W$, the Weyl group, and $\bar{\mathcal{O}_w}$ is the union of all $\mathcal{O}_v$ for $v \leq w$ in the Bruhat order

Want to study singularities of $\bar{\mathcal{O}_w}$, as measured by $IH^{2i}_x(\bar{\mathcal{O}_w}, \mathbb{Q})$

Kazhdan-Lusztig: given $v, w \in W$ with $v \leq w$, constructed $P_{v,w} \in \mathbb{N}[q]$ with

$$(\text{coef. of } q^i \text{ in } P_{v,w}) = \dim IH^{2i}_x(\bar{\mathcal{O}_w}, \mathbb{Q})$$

if $x \in \mathcal{O}_v$; all IH^{2i+1}_x are 0.

Upper semicontinuity: if $v \leq v' \leq w$, then

$$(\text{coef. of } q^i \text{ in } P_{v',w} \leq (\text{coef. of } q^i \text{ in } P_{v,w})$$

for all i; in words, the singularities of $\bar{\mathcal{O}}_w$ get worse as you go down. This terminology is due to Li and Yong; the result was proved by Irving, using ideas of Gabber and Joseph, but removing their hypothesis of the Jantzen conjecture. In particular, $\bar{\mathcal{O}}_w$ is rationally smooth if and only if $P_{1,w} = 1$: one only needs cohomology vanishing at a point, not in a neighborhood.

Now look at the symmetric case. K-orbits in G/B are not parametrized by W, but rather a closely related and typically much smaller set, e.g. if $G = GL(n), K = O(n)$, then these orbits are parametrized by just the involutions in $W = S_n$, ordered by the reverse Bruhat order.

Lusztig-Vogan: defined polynomials $P_{\mathcal{O},\mathcal{O}'} \in \mathbb{N}[q]$ indexesd by pairs of orbits $(\mathcal{O},\mathcal{O}')$ such that

$$(\text{coef. of } q^i \text{ in } P_{\mathcal{O},\mathcal{O}'}) = \dim IH_x^{2i}(\bar{\mathcal{O}}',\mathbb{Q}) \text{ for } x \in \mathcal{O}$$

and again $IH^{2i+1} = 0$.

Problem: upper semicontinuity fails badly. In fact the polynomials P depend on an ordered pair (γ,γ') of one-dimensional sheaves attached to orbits, not just on a pair of orbits. (This is not an issue in the Schubert variety case, as there all sheaves γ are trivial.) In general, if $\gamma_1,\gamma_2,\gamma_3$ are attached to the orbits $\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3$ and $\bar{\mathcal{O}}_1 \subseteq \bar{\mathcal{O}}_2 \subseteq \bar{\mathcal{O}}_3$, then we can have $P_{\gamma_2,\gamma_3} \neq 0, P_{\gamma_1,\gamma_3} = 0$. Thus even if one cares only about trivial sheaves, the presence of others causes difficulties.

Solution: assume away the problem, by assuming that all orbits $\mathcal{O}$ admit only the trivial sheaf, or equivalently that the real form G_0 of G corresponding to K has all Cartan subgroups connected.

Holds in several important cases, e.g.

$$G = GL(p+q), K = GL(p) \times GL(q), p \neq q$$

$$G = Sp(2p+2q), K = Sp(2p) \times Sp(2q)$$

$$G = SL(2n), K = Sp(2n)$$

Identify sheaves γ with their underlying orbits $\mathcal{O}$.

Then upper semicontinuity holds.

Sketch of proof: First note that by the recursion formulas of Vogan IC3, all $P_{\mathcal{O},\mathcal{O}'}$ have constant term 1 if $\bar{\mathcal{O}} \subseteq \bar{\mathcal{O}}'$. This is NOT a general fact (unlike the case of Schubert varieties), but in this setting it holds because all simple roots in the recursion formula are complex are type I noncompact imaginary.

Now we appeal to representation theory, returning for a moment to the setting of Verma modules. The Kazhdan-Lusztig conjectures imply that

$$P_{w_0 w, w_0 x}(1) = [M_w : L_x] \text{ for } x \leq w$$

where M_w, L_x denote respectively the Verma and simple highest modules of trivial infinitesimal character indexed by w and x, normalized so that $M_1 = L_1$ is irreducible, w_0 is the longest element of W, and the polynomials P are Kazhdan-Lusztig polynomials.

Gabber-Joseph and Irving refined this result as follows: define a series of submodules $\mathrm{soc}_i M_w$ inductively as follows:

$$\mathrm{soc}_0 M_w = \mathrm{soc}^0 M_w = 0$$

$$\mathrm{soc}_i M_w = \text{socle (sum of simple submodules) of } (M_w/\mathrm{soc}^{i-1} M_w)$$

$$\mathrm{soc}^i M_w = \text{preimage of } \mathrm{soc}_i M_w \text{ in } M_w$$

Then the soc_i form a filtration of M_w, of length equal to that of w in W, and

$$(\text{coef. of } q^i \text{ in } P_{w_0 x, w_0 y}) = [\mathrm{soc}_{\ell(y)+1+2i} M_w : L_y]$$

Now here in addition one has

$$w \leq w' \Leftrightarrow M_w \subseteq M'_w$$

by an old result of Verma. Upper semicontinuity than follows for Kazhdan-Lusztig polynomials by the definition of the socle filtration (given above).

Now we can extend all of this to the symmetric setting: under our hypotheses on G and K, Harish-Chandra modules for $(\mathfrak{g}, K)$ of trivial infinitesimal character (with $\mathfrak{g} =$ Lie G) form a single block $\mathcal{B}$. Pass to the dual block $\mathcal{B}'$ to $\mathcal{B}$ in the sense of Vogan IC4: this yields a new pair (G', K') of groups with G' complex reductive and K' a symmetric subgroup, such that there is a 1-1 order-reversing correspondence

$$(K\text{-orbits in } G/B) \leftrightarrow (\text{certain } K'\text{-orbits in } G'/B' \text{ and one-dimensional sheaves on them})$$

such that if $\mathcal{O}_1, \mathcal{O}_2$ are two K-orbits in G/B with $\bar{\mathcal{O}}_1 \subseteq \bar{\mathcal{O}}_2$ and if $\mathcal{O}_1', \mathcal{O}_2'$ are the corresponding K'-orbits in G'/B', parametrizing respectively standard and irreducible modules X_1', Y_2' for $(\mathfrak{g}', K')$, then

$$(\text{coef. of } q^i \text{ in } P_{\mathcal{O}_1,\mathcal{O}_2}) = \text{multiplicity of } Y_2' \text{ in an appropriate layer of the socle filtration of } X_1'$$

by a result of Casian and Collingwood, where standard modules are normalized to have unique irreducible quotients (not submodules), and we use that the socle filtration coincides with the weight filtration (which follows from Irving's results).

Now, corresponding to Verma's result cited above, I can show that whenever we have an inclusion $\bar{\mathcal{O}}'_1 \subseteq \bar{\mathcal{O}}'_2$ of orbit closures on the G' side, we get an inclusion $X_{\mathcal{O}'_1} \subseteq X_{\mathcal{O}'_2}$, using the action of wall-crossing operators on standard modules in $\mathcal{B}'$. The definition of the socle filtration then yields the desired upper semicontinuity for Kazhdan-Lusztig-Vogan polynomials corresponding to $\mathcal{B}$.

This result extends to the *principal* blocks of Harish-Chandra modules of trivial infinitesimal character if there is only conjugacy class of disconnected Cartan subgroups for the real form G_0 corresponding to (G, K) and the Cartan subgroups in this class have just two components. This covers the cases

$$(GL(2p), GL(p) \times GL(p))$$

$$(SO(2n), GL(n))$$

$$(E_7, E_6 \times \mathbb{C})$$

On the other hand, the pair $(G_2, SL(2) \times SL(2)$, corresponding to the split real form of type G_2, does not quite work, as here the disconnected Cartan subgroups have four components rather than two. Here upper semicontinuity holds for one block of Harish-Chandra modules (which has only one simple object), but fails for the pincipal block.

We conclude by mentioning that Braden and MacPherson have found a geometric proof of upper semicontinuity for Schubert varieties, which extends this property to a wide class of stratified varieties with a torus action. Their hypotheses rarely apply in the symmetric setting, however. It would be nice to find a more geometric argument in that setting.