Generalized Amitsur–Levitski Theorem and Equations for Sheets in a Reductive Complex Lie Algebra

Bertram Kostant, MIT

Representations of Reductive Groups

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Summary

My talk will connect various areas of Lie theory, polynomial identities, and representation theory.

I connect an old result of mine on a Lie algebra generalization of the Amitsur–Levitski Theorem with equations for sheets and tie this into recent results of Kostant–Wallach on the variety of singular elements in a reductive Lie algebra.

References

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1. The Amitsur–Levitski Theorem

Let me start with some results from the Amitsur–Levitski Theorem in reference [4].

Let R be an associative ring and for any $k \in \mathbb{Z}$ and $x_i, \ldots, x_k$, in R. One defines an alternating sum of products

$$[[x_1, \ldots x_k]] = \sum_{\sigma \in Sym\, k} sg(\sigma)\, x_{\sigma(1)} \cdots x_{\sigma(k)}.$$

One says that R satisfies the standard identity of degree k if $[[x_1, \ldots, x_k]] = 0$ for any choice of the $x_i \in R$. Of course, R is commutative if and only if it satisfies the standard identity of degree 2.

Now for any $n \in \mathbb{Z}$ and field F, let $M(n, F)$ be the algebra of $n \times n$ matrices over F. The following states the famous Amitsur–Levitski theorem.

Theorem 1.

$M(n, F)$ satisfies the standard identity of degree $2n$.

Remark 1. By restricting to matrix units, for a proof it suffices to take $F = \mathbb{C}$.

Without any knowledge that it was a known theorem, we came upon Theorem 1 (in my paper ref.[1]) a long time ago, from the point of Lie algebra cohomology. In fact, the result follows from the fact that if $\mathfrak{g} = M(n, \mathbb{C})$, then the restriction to $\mathfrak{g}$ of the primitive cohomology class of degree $2n+1$ of $M(n+1, \mathbb{C})$ to $\mathfrak{g}$ vanishes.

Of course $\mathfrak{g}_1 \subset \mathfrak{g}$, where $\mathfrak{g}_1 = \mathrm{Lie}\, SO(n, \mathbb{C})$. Assume n is even.

One proves that the restriction to $\mathfrak{g}_1$ of the primitive class of degree $2n-1$ (highest primitive class) of $\mathfrak{g}$ vanishes on $\mathfrak{g}_1$. This leads to a new standard identity, namely,

Theorem 2.

$$[[x_1, \ldots, x_{2n-2}]] = 0$$

for any choice of $x_i \in \mathfrak{g}_1$, *i.e., any choice of skew-symmetric matrices.*

Remark 2. Theorem 2 is immediately evident when $n = 2$.

Theorems 1 and 2 suggest that standard identities can be viewed as a subject in Lie theory. The next theorem offers support for this idea.

Let $\mathfrak{r}$ be a complex reductive Lie algebra and let

$$\pi : \mathfrak{r} \to \mathrm{End}\, V$$

be a finite-dimensional complex completely reducible representation. If $w \in \mathfrak{r}$ is nilpotent, then $\pi(w)^k = 0$ for some $k \in \mathbb{Z}$.

Let $\varepsilon(\pi)$ be the minimal integer k such that $\pi(w)^k = 0$ for all nilpotent $w \in \mathfrak{r}$.

In case π is irreducible, one can easily give a formula for $\varepsilon(\pi)$ in terms of the highest weight. If $\mathfrak{g}$ (resp. $\mathfrak{g}_1$) is given as above, and π (resp. π_1) is the defining representation, then $\varepsilon(\pi) = n$ and $\varepsilon(\pi_1) = n - 1$.

Consequently, the following theorem generalizes Theorems 1 and 2 (ref.[4]).

Theorem 3.

Let $\mathfrak{r}$ be a complex reductive Lie algebra and let π be as above. Then for any $x_i \in \mathfrak{r}$, $i = 1, \ldots, 2\varepsilon(\pi)$, one has

$$[[\hat{x}_1, \ldots, \hat{x}_{2\,\varepsilon(\pi)}]] = 0,$$

where $\hat{x}_i = \pi(x_i)$.

Henceforth $\mathfrak{g}$, until mentioned otherwise, will be an arbitrary reductive complex finite-dimensional Lie algebra. Let $T(\mathfrak{g})$ be the tensor algebra over $\mathfrak{g}$ and let $S(\mathfrak{g}) \subset T(\mathfrak{g})$ resp. $A(g) \subset T(\mathfrak{g})$) be the subspace of symmetric (resp. alternating) tensors in $T(\mathfrak{g})$. The natural grading on $T(\mathfrak{g})$ restricts to a grading on $S(\mathfrak{g})$ and $A(\mathfrak{g})$.

In particular, where multiplication is tensor product one notes the following:

Proposition 1.

$A^j(\mathfrak{g})$ is the span of $[[x_1, \ldots, x_j]]$ over all choices of x_i, $i = 1, \ldots, j$, in $\mathfrak{g}$.

Now let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then $U(\mathfrak{g})$ is the quotient algebra of $T(\mathfrak{g})$ so that there is an algebra epimorphism

$$\tau : T(\mathfrak{g}) \to U(\mathfrak{g}).$$

Let $Z = \text{Cent U}(\mathfrak{g})$ and let $E \subset U(\mathfrak{g})$ be the graded subspace spanned by all powers $e^j, j = 1, \ldots$, where $e \in \mathfrak{g}$ is nilpotent.

In (ref. [2], Theorem 21), where tensor product identifies with multiplication, we proved

$$U(\mathfrak{g}) = Z \otimes E.$$

And, in [4],(Theorem 3.4.) we proved the following.

Theorem 4.

For any $k \in \mathbb{Z}$ one has

$$\tau(A^{2k}(\mathfrak{g}) \subset E^k.$$

Theorem 3 is then an immediate consequence of Theorem 4.

Indeed, using the notation of Theorem 3, let $\pi_U : U(\mathfrak{g}) \to \mathrm{End}\ V$ be the algebra extension of π to $U(\mathfrak{g})$. One then has

Theorem 5.

If $E^k \subset \operatorname{Ker} \pi_U$*, then*

$$[[\hat{x}_1, \ldots, \hat{x}_{2k}]] = 0$$

for any $x_i, \ldots, x_{2k}$ *in* $\mathfrak{g}$.

The Poincaré–Birkhoff–Witt theorem says that the restriction $\tau : S(\mathfrak{g}) \to U(\mathfrak{g})$ is a linear isomorphism.

Consequently, given any $t \in T(\mathfrak{g})$, there exists a unique element $\bar{t}$ in $S(\mathfrak{g})$ such that

$$\tau(t) = \tau(\bar{t}).$$

Let $A^{even}(\mathfrak{g})$ be the span of alternating tensors of even degree. Restricting to $A^{even}(\mathfrak{g})$, one has a $\mathfrak{g}$-module map

$$\Gamma_T : A^{even}(\mathfrak{g}) \to S(\mathfrak{g})$$

defined so that if $a \in A^{even}(\mathfrak{g})$, then

$$\tau(a) = \tau(\Gamma_T(a)).$$

Now the (commutative) symmetric algebra $P(\mathfrak{g})$ over g and exterior algebra $\wedge\mathfrak{g}$ are quotient algebras of $T(\mathfrak{g})$. The restriction of the quotient map clearly induces $\mathfrak{g}$-module isomorphisms

$$\tau_S : S(\mathfrak{g}) \to P(\mathfrak{g})$$

$$\tau_A : A^{even}(\mathfrak{g}) \to \wedge^{even}\mathfrak{g}$$

where $\wedge^{even}\mathfrak{g}$ is the commutative subalgebra of $\wedge\mathfrak{g}$ spanned by elements of even degree.

We may complete the commutative diagram defining

$$\Gamma : \wedge^{even}\mathfrak{g} \to P(\mathfrak{g})$$

so that on $A^{even}(\mathfrak{g})$ one has

$$\tau_S \circ \Gamma_T = \Gamma \circ \tau_A.$$

Since we have shown that $U(\mathfrak{g}) = Z \otimes E$, one notes that for $k \in \mathbb{Z}$, one has

$$\Gamma : \wedge^{2k}\mathfrak{g} \to P^k(\mathfrak{g}).$$

The Killing form extends to a nonsingular symmetric bilinear form on $P(\mathfrak{g})$ and $\wedge\mathfrak{g}$. This enables us to identify $P(\mathfrak{g})$ with the algebra of polynomial functions on $\mathfrak{g}$ and to identify $\wedge\mathfrak{g}$ with its dual space $\wedge\mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual space to $\mathfrak{g}$.

2. Sheets

Let $R^k(\mathfrak{g})$ be the image of $\Gamma : \wedge^{2k}\mathfrak{g} \to P^k(\mathfrak{g})$
so that $R^k(\mathfrak{g})$ is a $\mathfrak{g}$-module of homogeneous polynomial functions of degree k on $\mathfrak{g}$.

The significance of $R^k(\mathfrak{g})$ has to do with the dimensions of $\mathrm{Ad}\,\mathfrak{g}$ adjoint (= coadjoint) orbits. Any such orbit is symplectic and hence is even dimensional.

For $j \in \mathbb{Z}$, let $\mathfrak{g}^{(2j)} = \{x \in \mathfrak{g} \mid \dim\,[\mathfrak{g}, x] = 2j\}$.

We recall that a $2j$ $\mathfrak{g}$-sheet is an irreducible component of $\mathfrak{g}^{(2j)}$.
Let $\mathrm{Var}\,R^k(\mathfrak{g}) = \{x \in \mathfrak{g} \mid p(x) = 0, \ \forall p \in R^k(\mathfrak{g})\}$.

Theorem 6 [see[4] Prop. 3.2.]

One has

$$\mathrm{Var}\, R^k(\mathfrak{g}) = \cup_{2j<2k}\, \mathfrak{g}^{(2j)},$$

or $\mathrm{Var}\, R^k(\mathfrak{g})$ *is the set of all* $2j$ $\mathfrak{g}$*-sheets for* $j < k$.

Let γ be the transpose of Γ. Thus

$$\gamma : P(\mathfrak{g}) \to \wedge^{even}\mathfrak{g},$$

and one has for $p \in P(\mathfrak{g})$ and $u \in \wedge\mathfrak{g}$,

$$(\gamma(p), u) = (p, \Gamma(u)).$$

One also notes

$$\gamma : P^k(\mathfrak{g}) \to \wedge^{2k}\mathfrak{g}.$$

A proof of Theorem 6 depends on establishing some nice algebraic properties of γ. Since we have, via the Killing form, identified $\mathfrak{g}$ with its dual, then $\wedge\mathfrak{g}$ is the underlying space for a standard cochain complex $(\wedge\mathfrak{g}, d)$ where d is the coboundary operator of degree $+1$.

In particular if $x \in \mathfrak{g}$, then $dx \in \wedge^2\mathfrak{g}$.

Identifying $\mathfrak{g}$ here with $P^1(\mathfrak{g})$, one has a map

$$P^1(\mathfrak{g}) \to \wedge^2\mathfrak{g}$$

.

Theorem 7.

The map $\gamma : P(\mathfrak{g}) \to \wedge^{even}\mathfrak{g}$ *is the homomorphism of commutative algebras extending* $P^1(\mathfrak{g}) \to \wedge^2\mathfrak{g}$.

In particular, for any $x \in \mathfrak{g}$,

$$\gamma(x^k) = (-dx)^k.$$

The connection with Theorem 6 follows next.

Proposition 2.

Let $x \in \mathfrak{g}$. Then $x \in \mathfrak{g}^{(2k)}$ if and only if k is maximal, such that $(dx)^k \neq 0$, in which case there is a scalar $c \in \mathbb{C}^\times$ such that

$$(dx)^k = c\ w_1 \wedge \cdots \wedge w_{2k},$$

where w_i, $i = 1, \ldots, 2k$, is a basis of $[x, \mathfrak{g}$.

Proofs of Theorem 7 and Proposition 2 are given in [ref. [4], as Theorem 1.4 and Proposition 1.3].

Now we wish to explicitly describe the $\mathfrak{g}$-module $R^k(\mathfrak{g})$. Details are in ref [4] Section 1.2.

Let $J = P(\mathfrak{g})^{\mathfrak{g}}$ so that J is the ring of $\mathrm{Ad}\,\mathfrak{g}$ polynomial invariants. Let $\mathrm{Diff}\, P(\mathfrak{g})$ be the algebra of differential operators on $P(\mathfrak{g})$ with constant coefficients.

One then has an algebra isomorphism

$$P(\mathfrak{g}) \to \mathrm{Diff}\ P(\mathfrak{g}), \quad q \mapsto \partial_q$$

where for $p, q, f \in P(\mathfrak{g})$, one has

$$(\partial_q p, f) = (p, qf)$$

and ∂_x, for $x \in \mathfrak{g}$, is the partial derivative defined by x.

Let $J_+ \subset J$ be the J-ideal of all $p \in J$ with zero constant term and let

$$H = \{q \in P(\mathfrak{g}) \mid \partial_p q = 0 \ \forall p \in J_+\}.$$

H is a graded $\mathfrak{g}$-module whose elements are called harmonic polynomials. Then one knows (see ref.[2], Theorem 11) that,

where the tensor product is realized by polynomial multiplication,

$$P(\mathfrak{g}) = J \otimes H.$$

It is immediate from $(\partial_q p, f) = (p, qf)$ that H is the orthocomplement of the ideal $J_+ P(\mathfrak{g})$ in $P(\mathfrak{g})$.

However since γ is an algebra homomorphism, one has

$$J_+ P(\mathfrak{g}) \subset \mathrm{Ker}\,\gamma$$

since one easily has that $J_+ \subset \mathrm{Ker}\,\gamma$.

This is clear since

$$\begin{aligned}\gamma(J_+) &\subset d(\wedge\,\mathfrak{g}) \cap (\wedge\,\mathfrak{g})^{\mathfrak{g}} \\ &= 0.\end{aligned}$$

But then $(\gamma(p), u) = (p, \Gamma(u))$ implies the following theorem.

Theorem 8.

For any $k \in \mathbb{Z}$ *one has*

$$R^k(\mathfrak{g}) \subset H.$$

Let $\mathrm{Sym}(2k, 2)$ be the subgroup of the symmetric group Sym(2k) defined by

$\mathrm{Sym}(2k, 2) = \{\sigma \in \mathrm{Sym}(2k) \mid \sigma$ permutes the set of

unordered pairs $\{(1, 2), (3, 4), \ldots, ((2k-1), 2k)\}\}$. That is, if

$\sigma \in \mathrm{Sym}(2k, 2)$ and $1 \leq i \leq k$, there exists $1 \leq j \leq k$, such that as unordered sets

$$(\sigma(2i-1), \sigma(2i)) = ((2j-1), 2j).$$

It is clear that $\mathrm{Sym}(2k, 2)$ is a subgroup of order $2^k \cdot k!$. Let $\Pi(k)$ be a cross-section of the set of left cosets of $\mathrm{Sym}(2k, 2)$ in $\mathrm{Sym}(2k)$ so that one has a disjoint union

$$\mathrm{Sym}(2k) = \cup \, \nu \, \mathrm{Sym}(2k, 2)$$

indexed by $\nu \in \Pi(k)$.

Remark 3. One notes that the cardinality of $\Pi(k)$ is $(2k-1)(2k-3)\cdots 1$ and the correspondence

$$\nu \mapsto ((\nu(1), \nu(2)), (\nu(3), \nu(4)), \ldots, (\nu((2k-1)), \nu(2k)))$$

sets up a bijection of $\Pi(k)$ with the set of all partitions of $(1, 2, \ldots, 2k)$ into a union of subsets each of which has two elements.

We also observe that $\Pi(k)$ may be chosen,and will be chosen, such that $sg\,\nu = 1$ for all $\nu \in \Pi(k)$.

This is clear since the sg character is not trivial on $\mathrm{Sym}(k, 2)$ for $k \geq 1$.

The following is a restatement of some results in [4], Section 3.2, especially (3.25) and (3.29).

Theorem 9.

For any $k \in \mathbb{Z}$ there exists a nonzero scalar c_k, such that for any x_i $i = 1, \ldots, 2k$, in $\mathfrak{g}$

$$\Gamma(x_1 \wedge \cdots \wedge x_{2k}) = c_k \sum_{\nu \in \Pi(k)} [x_{\nu(1)}, x_{\nu(2)}] \cdots [x_{\nu(2k-1)}, x_{\nu(2k)}].$$

Furthermore, the homogeneous polynomial of degree k on the right side of the equation above is harmonic, and $R^k(\mathfrak{g})$ is the span of all such polynomials for an arbitrary choice of the x_i.

We now come to the next section.

3. The Case $\mathfrak{h} = R$

Let $\mathfrak{h}$ be a Cartan sublgebra of $\mathfrak{g}$ and let $\ell = \dim \mathfrak{h}$, so $\ell = \operatorname{rank} \mathfrak{g}$.

Let Δ be the set of roots of $(\mathfrak{h}, \mathfrak{g})$ and let $\Delta_+ \subset \Delta$ be a choice of positive roots.

Let $r = \operatorname{card} \Delta_+$, so that $n = \ell + 2r$, where we fix $n = \dim \mathfrak{g}$.

We assume a well ordering is defined on Δ_+. For any $\varphi \in \Delta$, let e_φ be a corresponding root vector. The choices will be normalized only insofar as $(e_\varphi, e_{-\varphi}) = 1$ for all $\varphi \in \Delta$.

From Proposition 2 stated earlier, one recovers the well-known fact that $\mathfrak{g}^{(2k)} = 0$ for $k > r$, and $\mathfrak{g}^{(2r)}$ is the set of all regular elements in $\mathfrak{g}$.

One also notes then that the earlier statement $(\gamma(p), u) = (p, \Gamma(u))$ implies that $\mathrm{Var}\, R^r(\mathfrak{g})$ reduces to 0 if $k > r$, whereas Theorem 6 implies that

$$\mathrm{Var}\, R^r(\mathfrak{g}) \text{ is the set of all singular elements in } \mathfrak{g}.$$

The paper (ref [5] (joint with Nolan Wallach) is mainly devoted to a study of a special construction of $R^r(\mathfrak{g})$ and a determination of its remarkable $\mathfrak{g}$-module structure.

It is a classic theorem of C. Chevalley that J is a polynomial ring in ℓ homogeneous generators p_i, so that we can write

$$J = \mathbb{C}[p_1, \ldots, p_\ell].$$

Let $d_i = \deg p_i$. Then if we put $m_i = d_i - 1$, the m_i are referred to as the exponents of $\mathfrak{g}$, and one knows that

$$\sum_{i=1}^{\ell} m_i = r.$$

Now, henceforth assume $\mathfrak{g}$ is simple, so that the adjoint representation is irreducible. Let y_j, $j = 1, \ldots, n$, be the basis of $\mathfrak{g}$.

One defines an $\ell \times n$ matrix $Q = Q_{ij}$, $i = 1, \ldots, \ell$, $j = 1, \ldots, n$ by putting

$$Q_{ij} = \partial_{y_j} p_i.$$

Let S_i, $i = 1, \ldots, \ell$, be the span of the entries of Q in the i^{th} row.

The following proposition is immediate.

Proposition 3.

See ref [5].
$S_i \subset P^{m_i}(\mathfrak{g})$. *Furthermore* S_i *is stable under the action of* $\mathfrak{g}$ *and, as a* $\mathfrak{g}$*-module,* S_i *transforms according to the adjoint representation.*

If V is a $\mathfrak{g}$-module, let V_{ad} be the set of all of vectors in V which transform according to the adjoint representation. The equality $\mathrm{Sym}(2k) = \cup\, \nu\, \mathrm{Sym}(2k, 2)$ readily implies that $P(\mathfrak{g})_{\mathrm{ad}} = J \otimes H_{\mathrm{ad}}$.

Sometime ago I proved the following result—[See [2], Section 5.4. Especially, see (5.4.6) and (5.4.7).]

Theorem 10.

The multiplicity of the adjoint representation in H_{ad} is ℓ. Furthermore the invariants p_i can be chosen so that $S_i \subset H_{\mathrm{ad}}$ for all i and the $S_i, i = 1, \dots, \ell$, are indeed the ℓ occurrences of the adjoint representation in H_{ad}.

Clearly there are $\binom{n}{\ell}$ $\ell \times \ell$ minors in the matrix Q. The determinant of any of these minors is an element of $P^r(\mathfrak{g})$ by $\sum_{i=1}^{\ell} m_i = r$.

In [5] we offer a different formulation of $R^r(\mathfrak{g})$ by proving the following.

Theorem 11.

The determinant of any $\ell \times \ell$ minor of Q is an element of $R^r(\mathfrak{g})$ and indeed $R^r(\mathfrak{g})$ is the span of the determinants of all these minors.

The final section contains some additional results on the $\mathfrak{g}$-module structure of $R^r(\mathfrak{g})$.

We now show the $\mathfrak{g}$-module structure of $R^r(\mathfrak{g})$.

The adjoint action of $\mathfrak{g}$ on $\wedge\mathfrak{g}$ extends to $U(\mathfrak{g})$ so that $\wedge\mathfrak{g}$ is a $U(\mathfrak{g})$-module.

If $\mathfrak{s} \subset \mathfrak{g}$ is any subpace and $k = \dim \mathfrak{s}$, let $[\mathfrak{s}] = \wedge^k \mathfrak{s}$ so that $[\mathfrak{s}]$ is a 1-dimensional subspace of $\wedge^k \mathfrak{g}$.

Let $M_k \subset \wedge^k \mathfrak{g}$ be the span of all $[\mathfrak{s}]$, where $\mathfrak{s}$ is any k-dimensional commutative Lie subalgebra of $\mathfrak{g}$. If no such subalgebra exists, put $M_k = 0$. It is clear that M_k is a $\mathfrak{g}$-submodule of $\wedge^k \mathfrak{g}$.

Let $\mathrm{Cas} \in Z$ be the Casimir element corresponding to the Killing form. The following theorem was proved as Theorem (5) in [3].

Theorem 12.

For any $k \in \mathbb{Z}$ let m_k be the maximal eigenvalue of Cas on $\wedge^k \mathfrak{g}$. Then $m_k \leq k$.

Moreover $m_k = k$ if and only if $M_k \neq 0$ in which case M_k is the eigenspace for the maximal eigenvalue k.

Let Φ be a subset of Δ. Let $k = \mathrm{card}\,\Phi$ and write, in increasing order,

$$\Phi = \{\varphi_1, \ldots, \varphi_k\}.$$

Let

$$e_\Phi = e_{\varphi_1} \wedge \cdots \wedge e_{\varphi_k}$$

so that $e_\Phi \in \wedge^k \mathfrak{g}$ is an $(\mathfrak{h})$ weight vector with weight

$$\langle \Phi \rangle = \sum_{i=1}^{k} \varphi_i.$$

Let $\mathfrak{n}$ be the Lie algebra spanned by e_φ for $\varphi \in \Delta_+$, and let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$, defined by putting $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$.

Now a subset $\Phi \subset \Delta_+$ will be called an ideal in Δ_+ if the span, $\mathfrak{n}_\Phi$ of e_φ, for $\varphi \in \Phi$, is an ideal of $\mathfrak{b}$.

In such a case $\mathbb{C}e_\Phi$ is stable under the action of $\mathfrak{b}$ and hence if $V_\Phi = U(\mathfrak{g}) \cdot e_\Phi$, then where $k = \mathrm{card}\ \Phi$,

$$V_\Phi \subset \wedge^k \mathfrak{g}$$

is an irreducible $\mathfrak{g}$-module of highest weight $\langle \Phi \rangle$ having $\mathbb{C}e_\Phi$ as the highest weight space. We will say that Φ is abelian if $\mathfrak{n}_\Phi$ is an abelian ideal of $\mathfrak{b}$. Let

$$\mathcal{A}(k) = \{\Phi \mid \Phi \text{ be an abelian ideal of cardinality } k \text{ in } \Delta_+.\}$$

The following theorem was established in [3], (see especially Theorems (7) and (8).)

Theorem 13.

If Φ, Ψ are distinct ideals in Δ_+, then V_Φ and V_Ψ are inequivalent (i.e., $\langle\Phi\rangle \neq \langle\Psi\rangle$).

Furthermore if $M_k \neq 0$, then

$$M_k = \oplus_{\Phi \in \mathcal{A}(k)} V_\Phi$$

so that, in particular, M_k is a multiplicity 1 $\mathfrak{g}$-module.

We now focus on the case where $k = \ell$. Clearly $M_\ell \neq 0$ since $\mathfrak{g}^x$ is an abelian subalgebra of dimension ℓ for any regular $x \in \mathfrak{g}$.

Let $\mathcal{I}(\ell)$ be the set of all ideals of cardinality ℓ. The following theorem, giving the remarkable structure of $R^r(\mathfrak{g})$ as a $\mathfrak{g}$-module, is one of the main results in [5].

Theorem 14. *One has* $\mathcal{I}(\ell) = \mathcal{A}(\ell)$ *so that*

$$M_\ell = \oplus_{\Phi \in \mathcal{I}(\ell)} V_\Phi .$$

Moreover as $\mathfrak{g}$*-modules, one has the equivalence*

$$R^r(\mathfrak{g}) \cong M_\ell$$

so that $R^r(\mathfrak{g})$ *is a multiplicity* 1 $\mathfrak{g}$*-module with* $\mathrm{card}\, \mathcal{I}(\ell)$ *irreducible components and* Cas *takes the value* ℓ *on each and every one of the* $\mathcal{I}(\ell)$ *distinct components.*

Example. If $\mathfrak{g}$ is of type A_ℓ, then the elements of $\mathcal{I}(\ell)$ can be identified with Young diagrams of size ℓ. In this case, therefore the number of irreducible components in $R^r(\mathfrak{g})$ is $P(\ell)$, where P here is the classical partition function.