Elliptic Representations, Dirac Cohomology and Endoscopy

To David, with admiration

Jing-Song Huang

Department of Mathematics

Hong Kong University of Science and Technology

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Notations

- G, a connected semisimple Lie group.

  Θ, a Cartan involution.

  $K = G^{\Theta}$, the maximal compact subgroup.

- $\widehat{G}$, the unitary dual of G.

  $\mathfrak{g}_0 = \text{Lie}(G)$, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

  $\{\pi \in \widehat{G}\} \longleftrightarrow$ { Irreducible unitary $(\mathfrak{g}, K)$-modules X_π }.

- In later part of the talk, for the sake of elliptic representations, we turn to a linear algebraic real or p-adic group G.

  F, a real or p-adic field.

  $\mathbf{G}$, a connected semisimple linear algebraic group defined over F.

  $G = \mathbf{G}(F)$, the group of F-rational points on $\mathbf{G}$.

Dirac operators

- Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition. (Drop the subscript for the complexification.)
- $U(\mathfrak{g})$, the universal enveloping algebra.
  $Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$.
  $C(\mathfrak{p})$, the Clifford algebra w.r.t. the Killing form $B(\cdot,\cdot)|_{\mathfrak{p}}$.
- Let $Z_1, \cdots, Z_n$ be an orthogonal basis for $\mathfrak{p}_0$. Define

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

- D is indepedent of choice of bases, K-invariant, and satisfies

$$D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + C,$$

where Ω is the Casmir element and C is a constant.

Vogan's conjecture

- **Parthasarathy's Dirac Inequality:**
  $D\colon X_\pi \otimes S \to X_\pi \otimes S$ is self-adjoint. Thus,
  $\langle \Lambda, \Lambda \rangle \leq \langle \gamma + \rho_c, \gamma + \rho_c \rangle$, provided $X_\pi \otimes S(\gamma) \neq 0$.
- **Vogan's conjecture:** $\forall z \in Z(\mathfrak{g})$, $\exists\, \zeta(z) \in Z(\mathfrak{k}_\Delta)$, s.t.

  $$z \otimes 1 - \zeta(z) = Da + bD, \text{ for some } a, b \in U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$
- **Dirac cohomology** $H_D(X_\pi)\colon = \mathrm{Ker} D$ for unitary X_π;
  $H_D(X)\colon = \mathrm{Ker} D / \mathrm{Ker} D \cap \mathrm{Im} D$ for any $(\mathfrak{g}, K)$-module X.
- **Vogan Conjecture** implies: If $E_\gamma \subseteq H_D(X)$, then the infinitesimal character of X is conjugate to $\gamma + \rho_c$.
- **H-Pandzic (JAMS, 2002)** verified the Vogan's conjecture.

Kostant's cubic Dirac operator

- **Kostant (Proceedings of the Schur Conference, 2003)** Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ and $W_1, \ldots, W_l$ be an orthonormal basis of $\mathfrak{s}$.

$$D(\mathfrak{g}, \mathfrak{r}) = \sum_k W_k \otimes W_k - \frac{1}{2} \sum_{i<j<k} B([W_i, W_j], W_k) \otimes W_i W_j W_k.$$

- **Theorem** There is an $\zeta \colon Z(\mathfrak{g}) \to Z(\mathfrak{r}_\Delta)$, s.t. $\forall z \in Z(\mathfrak{g})$,
  $z \otimes 1 - \zeta(z) = Da + aD$, for some $a \in U(\mathfrak{g}) \otimes C(\mathfrak{s})$.

  Moreover, ζ is determined by

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\ \zeta\ } & Z(\mathfrak{r}) \\ \text{H.-C. isom}\big\downarrow & & \big\downarrow\text{H.-C. isom} \\ S(\mathfrak{h})^W & \xrightarrow{\text{Res}} & S(\mathfrak{h}_\mathfrak{r})^{W_\mathfrak{r}} \end{array}$$

- The infl'l characters of X and $H_D(X)$ are conjugate.

Dirac cohomology in other setting

- **Alekseev-Meinrenkein:** ‘Lie theory and the Chern-Weil homomorphism’ (Ann. Ecole. Norm. Sup. 2005)
- **Kumar:** ‘Induction functor in non-commutative equivariaint cohomology and Dirac cohomology’ (J. Algebra 2005)
- **H-Pandzic:** the symplectic Dirac operator in Lie superalgebras (Transf. Groups 2005)
- **Kac-Frajria-Papi:** the affine cubic Dirac operator in the affine Lie algebras (Adv. Math. 08)
- **Barbasch-Ciubotaru-Trapa:** the graded affine Hecke algebras (Acta Math. 2012)
- **Ciubotaru-He:** Weyl groups in connection with the Springer theory (Arkiv Math. 2013)

Calculation of Dirac cohomology

- In **H-Kang-Pandzic (Tran Group, 2009)**
  Let $\mathfrak{t} \subset \mathfrak{h}$ be the Cartan subalgebras of $\mathfrak{k}$ and $\mathfrak{g}$.
  Let $W(\mathfrak{g}, \mathfrak{t})$ be the Weyl group for the root system $\Delta(\mathfrak{g}, \mathfrak{t})$.
  Set $W^1(\mathfrak{g}, \mathfrak{t}) = \{w \in W(\mathfrak{g}, \mathfrak{t}) \mid w\rho$ is $\Delta^+(\mathfrak{k}, \mathfrak{t})$-dominant$\}$.
  Then $W(\mathfrak{g}, \mathfrak{t}) = W(\mathfrak{k}, \mathfrak{t}) \times W(\mathfrak{g}, \mathfrak{t})^1$. Set $l_0 = \mathrm{rank}\, \mathfrak{g} - \mathrm{rank}\, \mathfrak{k}$.
- **Theorem** Let V_λ be an irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ.
  If $\lambda \neq \Theta\lambda$, then $H_D(V_\lambda) = 0$.
  If $\lambda = \Theta\lambda$, then as a $\mathfrak{k}$ module,

$$H_D(V_\lambda) = \bigoplus_{w \in W(\mathfrak{g}, \mathfrak{t})^1} 2^{[l_0/2]} E_{w(\lambda+\rho)-\rho_c}.$$

- **Kostant:** cubic Dirac cohomology, equal rank case.
- **Mehdi-Zierau:** cubic Dirac cohomology, general case.

Dirac cohomology of HC modules

- In **H-Kang-Pandzic (Tran Group, 2009)**

  Let $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ be a θ-stable parabolic subalgebra.

  The $A_{\mathfrak{q}}(\lambda)$ is an admissible $(\mathfrak{g},K)$-module defined by cohomological parabolic induction from 1-dimensional $\mathfrak{l}$-module with parameter λ.

- **Theorem**
  If $\lambda\neq\theta\lambda$, then $H_D(A_{\mathfrak{q}}(\lambda))=0$.
  If $\lambda=\theta\lambda$, then

$$H_D(A_{\mathfrak{q}}(\lambda))=\bigoplus_{w\in W(\mathfrak{l},\mathfrak{t})^1}2^{[l_0/2]}E_{w(\lambda+\rho)-\rho_c}.$$

- **Mehdi-Parthasarathy (J. Lie Theory, 2011):** the generalized Enright-Varadarajan modules $B_{\mathfrak{p}}(\lambda)$

- **Barbasch-Pandzic:** certain unipotent representations

The $(\mathfrak{g}, K)$-cohomology

- In **H-Kang-Pandzic (Tran Group, 2009)**
  If $\dim \mathfrak{p}$ is even, then $\bigwedge^* \mathfrak{p} \cong S \otimes S^*$ as K-modules.
  If $\dim \mathfrak{p}$ is odd, then $\bigwedge^* \mathfrak{p}$ is 2-copies of $S \otimes S^*$.
  Consider the complex vector space $\mathrm{Hom}(\bigwedge^* \mathfrak{p}, X \otimes F^*)$.
  Then the complex of $H^*(\mathfrak{g}, K; X \otimes F^*)$ is
  $$\mathrm{Hom}_{\tilde{K}}(S \otimes S^*, X \otimes F^*) \cong \mathrm{Hom}_{\tilde{K}}(F \otimes S, X \otimes S).$$
  If X is unitary, **Wallach** has proved that the differential of this complex is 0. It follows that
- **Theorem** $H^*(\mathfrak{g}, K; X \otimes F^*) = \mathrm{Hom}_{\tilde{K}}(H_D(F), H_D(X))$.
- **Theorem (Vogan-Zuckerman, Comp Math, 1984)**
  $$\dim H^*(\mathfrak{g}, K; X \otimes F^*) = 2^{l_0} |W(\mathfrak{l}, \mathfrak{t}) / W(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})|.$$

Dirac cohomology in stages

- In **H-Pandzic-Renard (Repn Theory, 2006)**
  Let $\mathfrak{g} \supset \mathfrak{r} \supset \mathfrak{r}_1$.
  (i) $D(\mathfrak{g}, \mathfrak{r}_1) = D(\mathfrak{g}, \mathfrak{r}) + D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$;
  (ii) The summands $D(\mathfrak{g}, \mathfrak{r})$ and $D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$ anticommute.

- **Theorem** Let V be a unitary $(\mathfrak{g}, K)$-module. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$.
  Then
  $$H_D(\mathfrak{g}, \mathfrak{t}; V) = H_D(\mathfrak{k}, \mathfrak{t}; H_D(\mathfrak{g}, \mathfrak{k}; V)).$$
  $$H_D(\mathfrak{g}, \mathfrak{t}; V) = H(D(\mathfrak{g}, \mathfrak{k})\big|_{H_D(\mathfrak{k}, \mathfrak{t}; V)}).$$

- **Chuah-H (Crelle's J)** used calculation of Dirac cohomology in stages for study the geometric quantization of coadjoint orbits and construction of models of discrete series.

Lie algebra cohomology

- In **H-Pandzic-Renard (Repn Theory, 2006)**
  Let G be hermitian symmetric type. Recall that $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$. Then $\mathfrak{p}_0$ has a complex structure and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}^+ + \mathfrak{p}^-$.

- **Theorem** Assume that G is hermitian symmetric.
  Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra with $\mathfrak{l} \subset \mathfrak{k}$.
  If V is unitary, then

$$H_D(\mathfrak{g}, \mathfrak{l}; V) \cong H^*(\bar{\mathfrak{u}}, V) \otimes Z_{\rho(\bar{\mathfrak{u}})} \cong H_*(\mathfrak{u}, V) \otimes Z_{\rho(\bar{\mathfrak{u}})}.$$

- **Enright** calculated $H^*(\mathfrak{u}, V)$ for irreducible unitary highest weight modules V with $\mathfrak{l} = \mathfrak{k}$ and $\mathfrak{u} = \mathfrak{p}^+$ **(Crelle's J, 1988)**

- By Enright's result and calculation in stages, we obtained all three cohomologies in the above theorem.

Category $\mathcal{O}$

- In **H-Xiao (Selecta Math, 2012)**
  Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$.
  If $V \in \mathcal{O}^{\mathfrak{p}}$ is simple, then $H_D(V) \subset H^*(\bar{\mathfrak{u}}, V) \otimes Z_{\rho(\bar{\mathfrak{u}})}$.
  Actually, $H_D^{\pm}(\mathfrak{g}, \mathfrak{l}; V) \subset H^{\pm}(\bar{\mathfrak{u}}, V) \otimes Z_{\rho(\bar{\mathfrak{u}})}$.
- The parity condition is satisfied

$$\mathrm{Hom}_{\mathfrak{l}}(H^+(\bar{\mathfrak{u}}, V), H^-(\bar{\mathfrak{u}}, V)) = 0.$$

  It follows that

$$\mathrm{Hom}_{\mathfrak{l}}(H_D^+(V), H_D^-(V)) = 0.$$

- **Theorem** $V \in \mathcal{O}^{\mathfrak{p}}$ simple module.

$$H_D(\mathfrak{g}, \mathfrak{l}; V) \cong H^*(\bar{\mathfrak{u}}, V) \otimes Z_{\rho(\bar{\mathfrak{u}})} \cong H_*(\mathfrak{u}, V) \otimes Z_{\rho(\bar{\mathfrak{u}})}.$$

  Moreover, $H_D(\mathfrak{g}, \mathfrak{l}; V)$ is determined explicitly in terms of Kazhdan-Lusztig polynomials.

Applications

- In the monograph of **H-Pandzic (Birkhauser, 2006)**

1) **Parthasarathy (Ann Math, 1972)** geometric construction of most of discrete series.

2) **Atiyah-Schmid (Invent Math, 1977)** geometric construction of discrete series.

3) **Gross-Kostant-Ramond-Sternberg (PNAS, 1988)** generalized Weyl Character Formula.

4) **Kostant (Letters in Math Phys, 2000)** Generalized Bott-Borel-Weil Theorem.

5) **Langlands (A.J. Math, 1963)** the Langlands formula on multiplicity of automorphic forms.

- in **H-Pandzic-Zhu (A. J. Math, 2013)**

6) **Littlewood (PTRS 1944)** Littlewood Restriction Formulas.

7) **Enright-Willenbring (Ann Math, 2004)** generalized Littlewood Restriction Formulas.

Global characters

- Suppose (π, V) is an admissible representation of G and $f \in C_c^\infty(G)$. Then $\pi(f) = \int_G f(x)\pi(x)dx$ is of trace class. The global character of π is the distribution

$$f \to \hat{f}(\pi) = \mathrm{trace}\ (\pi(f)),\ f \in C_c^\infty(G).$$

- There is a locally integrable function Θ_π on G such that

$$\hat{f}(\pi) = \int_G f(x)\Theta_\pi(x)dx,\ f \in C_c^\infty(G).$$

- $\Theta_\pi(x)$ is real-analytic on the set G_{reg} of regular elements.

K-characters

- Suppose that $V = \sum_{i\in\widehat{K}} V_i$ is K-modules decomposition. The series $\Theta_K(V) = \sum_{i\in\widehat{K}} \Theta_K(V_i)$ converges to a distribution on K, and $\Theta_K(V) = \Theta_G(V)$ is real-analytic on $K \cap G_{\mathrm{reg}}$.
- Suppose that G has a compact Cartan subgroup T. Then $\dim \mathfrak{p}$ is even and $S = S^+ \oplus S^-$.

  $$0 \to \mathrm{Ker}D^+ \to X \otimes S^+ \to X \otimes S^- \to \mathrm{CoKer}D^+ \to 0.$$

  $$X \otimes S^+ - X \otimes S^- = \mathrm{Ker}D^+ - \mathrm{CoKer}D^+ = H_D^+(X) - H_D^-(X).$$
- $\Delta_{G/K}\Theta_G(V) = \mathrm{ch}H_D^+(X) - \mathrm{ch}H_D^-(X)$ on $K \cap G_{\mathrm{reg}}$. Here,

  $$\Delta_{G/K} = \mathrm{ch}S^+ - \mathrm{ch}S^- = \pm \prod_{\alpha\in\Delta_n^+(\mathfrak{g},\mathfrak{t})} (e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}).$$

Elliptic representations

- F, real or p-adic field. $\mathbf{G}$, is linear algebraic group defined over F. $G = G(F)$, the group of F-rational points.
- $D(\gamma) = \det(1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}$ Weyl discriminant.
- Note that $G_{\mathrm{reg}}(F) \cap G_{\mathrm{ell}}(F)$ is an open set in $G(F)$.
- π is elliptic if Θ_π does not vanish on $G_{\mathrm{reg}}(F) \cap G_{\mathrm{ell}}(F)$, i.e.,

  $$\Phi_\pi(\gamma) = |D(\gamma)|^{\frac{1}{2}} \Theta_\pi(\gamma) \neq 0, \text{ for some } \gamma \in G_{\mathrm{reg}}(F) \cap G_{\mathrm{ell}}(F).$$
- $\Delta_{G/K}(\gamma)\Theta_\pi(\gamma)$ and $\Phi_\pi(\gamma)$ has the same absolute value on $G_{\mathrm{reg}}(\mathbb{R}) \cap G_{\mathrm{ell}}(\mathbb{R})$.
- Set the Dirac index $\theta_\pi = \mathrm{ch} H_D^+(X_\pi) - \mathrm{ch} H_D^-(X_\pi)$. Then $(\Theta_\pi, \Theta_\pi)_{\mathrm{ell}} = (\theta_\pi, \theta_\pi)_K$. Here $(\,\cdot, \cdot\,)_{\mathrm{ell}}$ is defined in the next slide. Consequently, we get
- **Theorem** π is elliptic iff $\theta_\pi \neq 0$

Orthogonal relations

- The tempered elliptic representations satisfy the orthogonal relation w.r.t.

$$(\Theta_\pi, \Theta_{\pi'})_{\mathrm{ell}} = |W(G(\mathbb{R}), T_{\mathrm{ell}}(\mathbb{R}))|^{-1} \int_{T_{\mathrm{ell}}(\mathbb{R})} |D(\gamma)| \Theta_\pi(\gamma) \overline{\Theta_{\pi'}(\gamma)} d\gamma.$$

- **Theorem** If π, π' are discrete series representations, then

$$(\Theta_\pi, \Theta_{\pi'})_{\mathrm{ell}} = \delta(\pi, \pi')(: = \dim \mathrm{Hom}_G(\pi, \pi'))$$

- The above identity follows easily from

$$(\Theta_\pi, \Theta_{\pi'})_{\mathrm{ell}} = (\theta_\pi, \theta_{\pi'})_K = \langle \chi_\mu, \chi_{\mu'} \rangle.$$

Dirac index

- $G(\mathbb{R}) \supset K(\mathbb{R}) \supset T(\mathbb{R})$ of equal rank.
  V, a simple Harish-Chandra module.
- **Theorem** If V has regular infinitesimal character, then
  $\theta_V = 0$ iff $H_D(V) = 0$ (i.e. $\mathrm{Hom}_{\widetilde{K}}(H_D^+, (V), H_D^-(V)) = 0$).
- Let $\mathfrak{b} = \mathfrak{t} + \mathfrak{u}$ be a Θ-stable Borel subalgebra. Then
  $H_D^\pm(\mathfrak{g}, \mathfrak{t}; V) \subseteq H^\pm(\mathfrak{u}, V) \otimes Z_{\rho(\bar{\mathfrak{u}})}$.
  (It follows that $H_D^\pm(\mathfrak{g}, \mathfrak{t}; V) \cong H^\pm(\mathfrak{u}, V) \otimes Z_{\rho(\bar{\mathfrak{u}})}$.)
- **Vogan (Duke M. J., 1979)** (2nd in a series of 4 papers)

$$\mathrm{Hom}_T(H^+(\mathfrak{u}, V), H^-(\mathfrak{u}, V)) = 0.$$

  Then the above parity condition follows from Vogan's theorem and the calculation in stages.
- **Conjecture:** For any irreducible π, $\theta_\pi \neq 0$ iff $H_D(X_\pi) \neq 0$.
- The above conjecture holds if π is tempered.

Supertempered distributions

- In the last paper of **Harish-Chandra's Collected Papers**
- $G = G(\mathbb{R}) \supset K(\mathbb{R}) \supset T(\mathbb{R})$ of equal rank.
- **Theorem** For $\mu \in \widehat{T(\mathbb{R})}$, there is a unique supertempered distribution Θ_μ, s.t.

$$\Delta\Theta_\mu(\gamma) = \sum_{w \in W_K} \epsilon(w) e^{w\mu}.$$

- If π is tempered and elliptic, then Θ_π is supertemperred.
- **Theorem** If π_1, π_2 are irreducible tempered elliptic representations, then either $(\Theta_{\pi_1}, \Theta_{\pi_2})_{\mathrm{ell}} = 0$ or $\Phi_{\pi_1} = \pm\Phi_{\pi_2}$.
- Consequently, the discrete series together with some of the limit of discrete series form an orthonomal basis of the space of supertempered distributions.

Elliptic tempered characters

- In **Arthur (Acta Math, 1993)**
- A description of classification of irreducible elliptic tempered representations for real and p-adic groups.
- Invariant elliptic orbital integrals are dual to the tempered elliptic representations.
- There is another basis consisting of virtual elliptic characters, which is convenient for studying Fourier transform of the weight orbital integrals. **Arthur (Crelle's J, 1994)**
- Elliptic representations are important for studying the trace formulas and automorphic forms.

Regular infinitesimal characters

- Can we classify unitary elliptic representations of $G(\mathbb{R})$? (or classifying irreducible unitary representations with nonzero Dirac cohomology.)
- **Salamanca-Riba (Duke M. J. 1998)** If X is an irreducible unitary $(\mathfrak{g}, K)$-module with strongly regular infinitesimal character, then $X \cong A_{\mathfrak{q}}(\lambda)$.
- **Theorem** Let G be a connected linear algebraic semisimple Lie group with a compact Cartan subgroup. Suppose π is an irreducible elliptic representation of G with a regular infinitesimal character. Then $X_\pi \cong A_{\mathfrak{q}}(\lambda)$.
- **Theorem** Let π_1, π_2 be representations in above setting. Then $X_\pi \cong X_{\pi'}$ iff $H_D(X_\pi) = H_D(X_{\pi'})$.
- The above statements are false if the condition on infinitesimal character fails.

Orbital integrals

- The orbital integrals are parametrized by the set of regular semisimple conjugacy classes in G.

$G_{\mathrm{reg}} = \{\gamma \in G \mid \gamma$ semisimple, the eigenvalues are distinct$\}$.

- Orbital integral $\mathcal{O}_\gamma(f) = \int_{G/G_\gamma} f(x^{-1}\gamma x)dx, f \in C_c^\infty(G)$.
- Stable orbital integral $S\mathcal{O}_\gamma(f) = \sum_{\gamma' \in S(\gamma)} \mathcal{O}_\gamma(f)$.
Here $S(\gamma)$ is the stable conjugacy class.

Pseudo-coefficients

- In **Labesse (Math Ann, 1991)**
  Let π be a discrete series representation with Dirac cohomology E_μ (HC parameter is $\mu + \rho_c$).
  Set $\theta_{\mathbb{1}} = \mathrm{ch} H_D^+(\mathbb{1}) - \mathrm{ch} H_D^-(\mathbb{1}) = \mathrm{ch} S^+ - \mathrm{ch} S^-$.
- Set $f_\pi = \theta_{\mathbb{1}} \cdot \chi_\mu$
  Then $(f_\pi, \Theta_{\mu'})_{\mathrm{ell}} = (\chi_\mu, \chi_{\mu'}) = \dim \mathrm{Hom}_K(E_\mu, E_{\mu'})$.

  So f_π is a pseudo-coefficient for π.
- $\mathcal{O}_\gamma(f_\pi) = \Theta(\gamma^{-1})$ if γ is elliptic.
  $\mathcal{O}_\gamma(f_\pi) = 0$ if γ is not elliptic.

Endoscopic transfer

- This is established by **Shelstad** in a series of papers.
- Assume that $G(\mathbb{R})$ has a compact Caratn subgroup $T(\mathbb{R})$. Let κ be an endoscopic character defining an endoscopic group H.
- **Labesse (AMS, 2006)** calculated $f \to f^H$ so that

$$SO_{\gamma_H}(f_\mu^H) = \Delta(\gamma_H, \gamma_G)\mathcal{O}^\kappa_{\gamma_G}(f_\mu)$$

  for the pseudo-coefficients of discrete series f_μ. Here $f_\mu^H = \sum_{w \in W(\mathfrak{g})/W(\mathfrak{h})} a(w,\mu) g_{w\mu}$ with $a(w,\mu)$ depending on κ, and $g_{\mu'}$ are pseudo-coefficients of discrete series for H.
- The discrete series L-packets are in bijection with the irreducible finite-dimensional representations of the same infinitesimal character.

The Arthur packets

- For the non-tempered case, we need Arthur packets Π_ψ.
- In **Adams-Johnson (Comp Math, 1987)**, they constructed packets of non-tempered representations. Set $S = W_K \backslash W(\mathfrak{g}, \mathfrak{t})/W(\mathfrak{l}, \mathfrak{t})$. Identify π with its character Θ_π.
- **Theorem** $\sum_{w\in S} \epsilon(w)A(w\lambda)$ is stable. Here $A(w\lambda) = A_{\mathfrak{q}}(w\lambda)$ with $\mathfrak{q}$ depending on w.
- Let $(\kappa, \mathbf{H})$ be an endoscopic group of $\mathbf{G}$ which contains a group isomorphic to $\mathbf{L}$.
- **Theorem**

$$\mathrm{Lift} \sum_{w\in S'} \epsilon(w)A(w\lambda') = \pm \sum_{w\in S} \epsilon(w)\kappa(w)A(w\lambda).$$

- **Theorem** If $f \mapsto f^H$ and $\Theta = \mathrm{Lift}_H^G \Theta'$, then $\Theta(f) = \Theta'(f^H)$.

The fundamental lemma

- The fundamental lemma is conjectured by **Langlands**.

$$SO_{\gamma_H}(\mathbb{1}_{K_H}) = \Delta(\gamma_H, \gamma_G)\mathcal{O}^{\kappa}_{\gamma_G}(\mathbb{1}_K).$$

- It was proved by **Shelstad (Math Ann, 1982)** for $G(\mathbb{R})$.
- The progress was made by **Waldspurger, Laumon, Ngo**.
- It was proved by **Ngo (IHES, 2010)** for p-adic groups.
- Recall **Labesse** calculated the transfer of the pseudo-coefficients $f_\mu = \theta_{\mathbb{1}} \cdot \chi_\mu$.
- Let $\pi = A_{\mathfrak{b}}(\lambda)$ ($\lambda = -\rho_n$) be a limit of discrete series so that the Dirac index of π is equal to $\theta_{\mathbb{1}} \cdot \Theta_K(\pi) = \mathbb{1}_K$. By the Blattner's formula, one has decomposition $\Theta_K(\pi) = \sum_\mu m_\mu \chi_\mu$.
- It is an interesting question to see how to match two sides in the fundamental lemma by Labesse's calculation.

Hypo-elliptic representations

- $G(\mathbb{R}) \supset K(\mathbb{R})$, not necessarily equal rank.
- A representation is called **hypo-elliptic** if its global character is not identically zero on the set of regular elements in a fundamental Cartan subgroup.
- **Conjecture:** If $\pi \in \widehat{G}$ and $H_D(X_\pi) \neq 0$, then π is hypo-elliptic.
- Recall that irreducible tempered representations are induced from tempered elliptic representations.
- **Conjecture:** a unitary representation is either having nonzero Dirac cohomology or induced from a unitary representation with nonzero Dirac cohomology.
- The conjecture holds for $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $GL(n,\mathbb{H})$ as well as $\widetilde{GL}(n,\mathbb{R})$ (the two-fold covering group of $GL(n,\mathbb{R})$).

David, Happy Birthday!

- **Thank you for guiding me into the field.**
- **Thank you for sharing your ideas and insights.**
- **Thank you for being a great teacher and friend.**