Newforms for odd orthogonal groups

Benedict Gross

Harvard University

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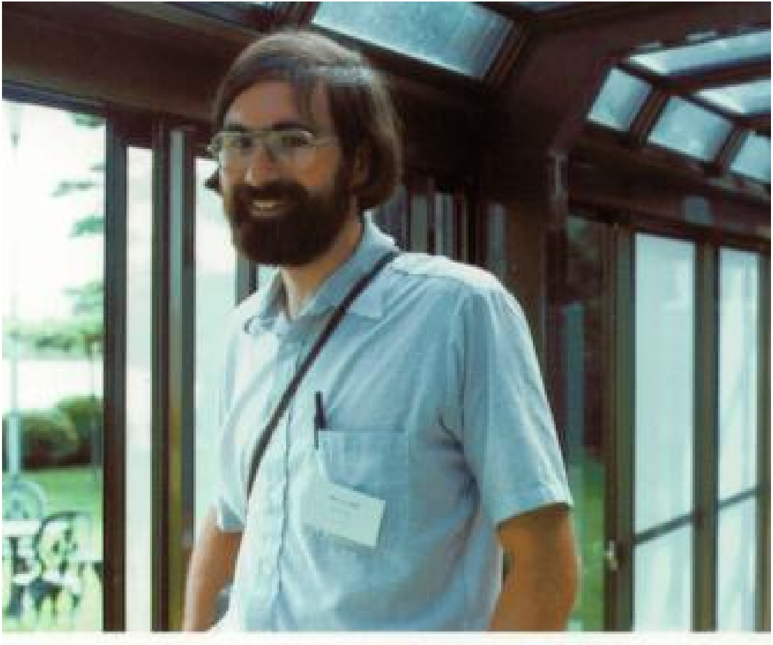


Let V be an orthogonal space of signature $(n+1, n)$ over $\mathbb{R}$.

Let $G = \mathrm{SO}(V)(\mathbb{R}) = \mathrm{SO}(n+1, n) \quad G/G^0 = \mathbb{R}^*/\mathbb{R}^{*2}$.

Fix a decomposition into definite subspaces $V = V_+ \oplus V_-$

$K = S(O(V_+) \times O(V_-)) \;\; K^0 = \mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$.

Let T be a maximal torus in K. The discrete series for G are parametrized by orbits of the compact Weyl group W_K on the set of regular elements $\alpha \in X^*(T) + \rho$.

A regular element α determines a root basis Δ for T.

The discrete series representation π_α is **generic** if and only if all of the roots in the basis Δ are **non-compact**.

Since W_K acts transitively on the generic chambers, we may assume that the generic discrete series representation π_α has parameter in the standard Weyl chamber, where the simple roots

$$\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n\}$$

are all non-compact.

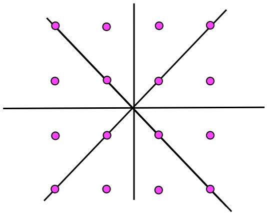
The parameter α then has the form

$$\alpha = (\alpha_1 > \alpha_2 > \ldots > \alpha_{n-1} > \alpha_n > 0)$$

where the α_i are all half integers.

A generic limit discrete series has parameter α in the closure of this chamber

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-1} \geq \alpha_n > 0)$$



The minimal K-type V_λ in the generic limit discrete series π_α has highest weight

$$\lambda = \alpha + \rho_n - \rho_c$$

Since $\rho_n - \rho_c = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ we have

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 1).$$

What does this mean?

Since the simple roots are all non-compact, the simple compact roots are

$$\{\epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_4, \epsilon_3 - \epsilon_5, \ldots, \epsilon_{n-2} - \epsilon_n, \epsilon_{n-2} + \epsilon_n, \epsilon_{n-1}\}.$$

This is the union of two orthogonal subsets, determined by the parity of the subscripts.

When $n = 6$, so $G = \mathrm{SO}(7, 6)$, the simple compact roots are

$$\{\epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_5, \epsilon_5\} \cup \{\epsilon_2 - \epsilon_4, \epsilon_4 - \epsilon_6, \epsilon_4 + \epsilon_6\}.$$

These are the simple roots for SO(7) and SO(6).

Since

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_n \geq 1).$$

the representation V_λ is induced from the representation $V_\mu \otimes V_\nu$ of the product $\mathrm{SO}(V_+) \times \mathrm{SO}(V_-)$, with

$$\mu = (\lambda_1 \geq \lambda_3 \geq \ldots) \quad \nu = (\lambda_2 \geq \lambda_4 \geq \ldots).$$

Since these weights are **interlaced**, we conclude from the classical branching formula that $V_\mu \otimes V_\nu$ has a **unique line** fixed by the diagonally embedded subgroup $\mathrm{SO}(V_-)$.

Passing to the induced representation V_λ, we have the following.

Theorem

Let π be a generic limit discrete series representation of $G = \mathrm{SO}(n+1, n)$.

Then there is a unique line in the minimal K-type which is fixed by the subgroup $H = O(n) \to K = S(O(n+1) \times O(n))$.

When $n = 1$,

$$G = \mathrm{SO}(2, 1) = \mathrm{PGL}(2, \mathbb{R}),\ K = O(2),\ H = O(1).$$

The minimal K-type of every discrete series has dimension 2.

When $n = 2m$ and $\lambda = (1, 1, 1, \ldots, 1)$,

$$V_\lambda = \wedge^m(V_+) \otimes \wedge^m(V_-) = \wedge^m(2m+1) \otimes \wedge^m(2m).$$

Is there an analogous result for generic representations of the split odd orthogonal group over $\mathbb{Q}_p$?

Let V be a vector space of dimension $2n+1$ over $\mathbb{Q}$ with basis $\{e_1, e_2, \ldots, e_n, v, f_n, f_{n-1}, \ldots, f_1\}$ and non-zero inner products

$$\langle e_i, f_i \rangle = 1,\ \langle v, v \rangle = 2.$$

$$A = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & 2 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}. \tag{1}$$

Let $G = \mathrm{SO}(V)$ over $\mathbb{Q}$. $\quad G(\mathbb{R}) = \mathrm{SO}(V \otimes \mathbb{R}) = \mathrm{SO}(n+1, n)$

Let $L(1)$ be the $\mathbb{Z}_p$-lattice in $V \otimes \mathbb{Q}_p$ spanned by

$$\{e_1, e_2, \ldots, e_n, v, f_n, f_{n-1}, \ldots, f_1\}$$

Let $K(1)$ be the hyperspecial maximal compact subgroup which stabilizes $L(1)$, with reduction $\mathrm{SO}(2n+1, p)$..

For $a \geq 1$ let $L(p^a)$ be the $\mathbb{Z}_p$-lattice in $V \otimes \mathbb{Q}_p$ spanned by

$$\{e_1, e_2, \ldots, e_n, p^a.v, p^a.f_n, p^a.f_{n-1}, \ldots, p^a.f_1\}$$

The scaled inner product $\frac{1}{p^a}\langle , \rangle$ is integral on $L(p^a)$.

Its radical (mod p) is spanned by the image of $p^a.v$.

Let $J(p^a)$ be the compact open subgroup which stablilzes the lattice $L(p^a)$. Then $J(p^a)$ has reduction $O(2n, p)$.

Let $K(p^a)$ be the subgroup of $J(p^a)$ which reduces to $\mathrm{SO}(2n, p)$.

Theorem (Pei-Yu Tsai)

Let π *be a generic irreducible representation of the p-adic group* $G(\mathbb{Q}_p) = \mathrm{SO}(V)(\mathbb{Q}_p)$.

There is a unique integer $f \geq 0$ *such that the open compact subgroup* $K(p^f)$ *fixes a one dimensional subspace of* π*. The fixed space of* $K(p^a)$ *is zero for all* $a < f$ *and has dimension greater than one for all* $a > f$.

The integer f *is the Artin conductor of the standard representation* M *of the Langlands parameter of* π*, which is a symplectic representation of dimension* $2n$ *of the Weil-Deligne group of* $\mathbb{Q}_p$.

When $f \geq 1$ *the action of the quotient group* $J(p^f)/K(p^f)$ *on the line fixed by* $K(p^f)$ *is given by the local epsilon factor* $\epsilon_p(M) = \pm 1$.

When $n = 1$, $G = \mathrm{PGL}(2)$, $K(p^a) = \Gamma_0(p^a)$, and generic = infinite dimensional.

In this case, the theorem was proved by W. Casselman in 1973.

When $n = 2$, $G = \mathrm{PGSp}(4)$ and $K(p^a)$ are the paramodular subgroups of B. Roberts and R. Schmidt.

The general definition of open compact subgroups was suggested by A. Brumer, and the proof of the theorem relies on an integral representation of the standard L-function $L(\pi_p, M, s)$ of a generic representation found by D. Soudry.

For the Steinberg representation $f = 2n - 1$ and $\epsilon_p = -1$.

For depth zero supercuspidal representations $f = 2n$.

For the generic limit discrete series over $\mathbb{R}$

$$L(\pi_\infty, M, s) = \prod \Gamma_{\mathbb{C}}(s + \lambda_i) \quad \epsilon_\infty(M) = (-1)^{\sum \lambda_i}.$$

Let M be a motive of weight -1 and rank $2n$ over $\mathbb{Q}$ with a non-degenerate symplectic pairing

$$M \times M \to \mathbb{Q}(1).$$

$M = H_1(X)$ for a curve X of genus n $\quad M = H^{2m-1}(X)(m)$

The ℓ-adic realizations M_ℓ give Galois representations

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}(M_\ell) = \mathrm{GSp}_{2n}(\mathbb{Q}_\ell).$$

If $M = H_1(X)$ then $M_\ell = T_\ell(J)$ with the Weil pairing.

The restriction to a decomposition group at the prime $p \neq \ell$ gives the Langlands parameter of a generic, irreducible representation $\pi_p(M)$ of $\mathrm{SO}(V)(\mathbb{Q}_p)$, whose conductor is equal to the Artin conductor of the Galois representation M_ℓ.

The Hodge realization gives n pairs $(p_i, q_i) + (q_i, p_i)$ with $p_i + q_i = -1$.

Assume that $p_i \geq 0$ and that $p_1 \geq p_2 \geq \ldots \geq p_n$.

$$\alpha = (\frac{p_1 - q_1}{2}, \frac{p_2 - q_2}{2}, \ldots, \frac{p_n - q_n}{2})$$

is the parameter of a generic limit discrete series $\pi_\infty(M)$ of $\mathrm{SO}(n+1, n)$, whose minimal K type has highest weight

$$\lambda = (p_1 + 1, p_2 + 1, \ldots, p_n + 1)$$

If $M = H_1(X)$ for a curve X, then all $(p, q) = (0, -1)$ and

$$\lambda = (1, 1, \ldots, 1).$$

Note that π_∞ is **not** in the discrete series when the genus of X satisfies $n \geq 2$.

Recall that $G = \mathrm{SO}(V)$ over $\mathbb{Q}$.

Since the irreducible representations $\pi_p(M)$ of $G(\mathbb{Q}_p)$ are unramified for almost all primes p, we may form the restricted tensor product over all places v of $\mathbb{Q}$

$$\pi(M) = \bigotimes \pi_v(M).$$

This is an irreductible generic representation of the adelic group $G(\mathbb{A})$, associated to the symplectic motive M.

Langlands conjectures that the representation $\pi(M)$ is automorphic, or more precisely that the complex vector space

$$Hom_{G(\mathbb{A})}(\pi(M), \mathcal{F}(G(\mathbb{Q})\backslash G(\mathbb{A})))$$

has dimension one. This would imply that the global L-function $L(M, s)$ has an analytic continuation and a functional equation.

We want to refine this conjecture, in the style of Weil's precise conjecture for elliptic curves over $\mathbb{Q}$, by fixing a line in the adelic representation $\pi(M) = \bigotimes \pi_v(M)$.

Let $N = \prod p^{f(p)}$ be the Artin conductor of the Galois representation M_ℓ.

Recall the basis $\{e_1, \ldots, e_n, v, f_n, \ldots, f_1\}$ of V, and let $L(N)$ be the lattice in V which is the $\mathbb{Z}$ span of the vectors $\{e_1, e_2, \ldots, N.v, N.f_n, \ldots, N.f_1\}$

For every prime p, the lattice $L(N) \otimes \mathbb{Z}_p = L(p^f)$ in $V \otimes \mathbb{Q}_p$ defines an open compact subgroup $K_p = K(p^f)$ of $G(\mathbb{Q}_p)$ which fixes a one dimensional subspace of $\pi_p(M)$.

We want to define a line in $\pi_\infty(M)$.

Let V_+ ve the positive definite subspace of $V \otimes \mathbb{R}$ spanned by the $n+1$ vectors $e_i + f_i$ and v, and let V_- be the negative definite subspace spanned by the n vectors $e_i - f_i$.

The decomposition $V = V_+ \oplus V_-$ defines the subgroup K_∞ of $G(\mathbb{R})$ and hence a subspace V_λ of $\pi_\infty(M)$.

Let θ be the anti-isometry $\theta : V_- \to V_+$ mapping $e_i - f_i$ to $e_i + f_i$.

This induces $\eta : O(V_-) \to O(V_+)$ defined by $\eta(g) = (\theta.g.\theta^{-1}, 1)$, where θ^{-1} is defined on the orthogonal complement of v.

We let H_∞ be the subgroup of K_∞ defined by the image of the diagonal embedding

$$O(V_-) \to K_\infty = S(O(V_+) \times O(V_-))$$

Then H_∞ fixes a line in V_λ.

Corollary

The compact subgroup $H_\infty \times \prod K_p$ of $G(\mathbb{A})$ is well-defined up to conjugacy by $G(\mathbb{Q})$ and fixes a distinguished line in the generic representation $\pi(M) = \bigotimes \pi_v(M)$ of $G(\mathbb{A})$.

The combination with Langlands's conjecture gives a **newform**

$$F : G(\mathbb{Q})\backslash G(\mathbb{A})/(H_\infty \times \prod K_p) \to \mathbb{C}$$

associated to the symplectic motive M.

The function F is only defined up to scaling.

$$a_1(F) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} F(u)\psi(u)du = 1$$

where $\psi : U(\mathbb{Q})\backslash U(\mathbb{A}) \to \mathbb{C}^*$ is an integral generic character.

The newform F is determined by its restriction to the real group

$$F_\infty : \Gamma_0(N)\backslash G(\mathbb{R})/H_\infty \to \mathbb{C},$$

where $\Gamma_0(N)$ is the arithmetic group determined by the lattice $L(N)$ in $V \otimes \mathbb{R}$.

There is a vector valued function

$$J : \Gamma_0(N)\backslash G(\mathbb{R}) \to V_\lambda$$

$$J(gk) = k.J(g)$$

The inner product with the H-invariant line in V_λ recovers F_∞.

For $n = 1$, $G = \mathrm{PGL}(2)$ and we get a harmonic newform:

$$H(\tau) = \sum_{n \geq 1} a_n(q^n + \bar{q}^n) = 2. \sum_{n \geq 1} a_n e^{-2\pi n y} cos(2\pi n x)$$

with $a_1 = 1$.

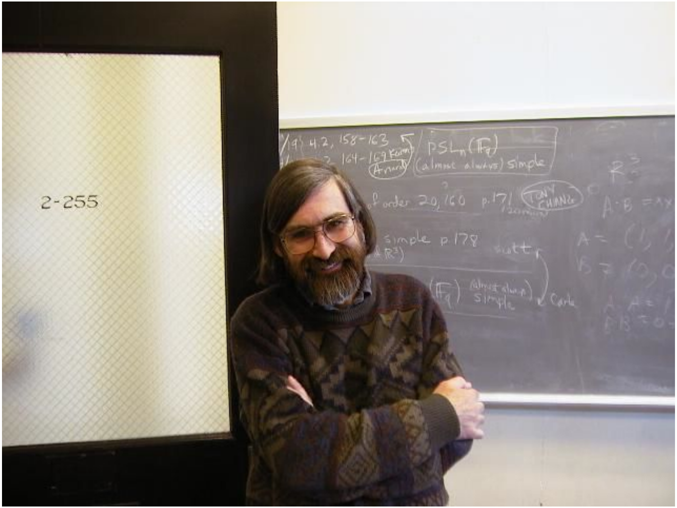
The coefficients a_n in this expansion are real numbers, so the function H takes real values.

This should also be true for the newform F_∞ on $G(\mathbb{R})$ with $a_1(F) = 1$.

The infinitesimal character α of $\pi_\infty(M)$ puts F_∞ in a finite dimensional real vector space, which should be explicitly computable.

The equivariant tangent bundle of the homogeneous space $\mathrm{SO}(n+1, n)/O(n)$ has a sub line bundle fixed by $O(n)$.

Thanks – and see you soon, David.



I'll miss your old office.