Computing with left cells of type E_8

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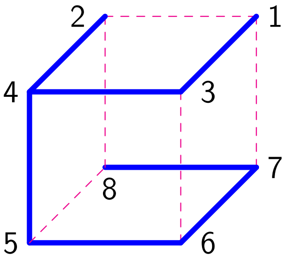
- The quantity $\frac{\dim \mathfrak{g}}{\mathrm{rank}(\mathfrak{g})^2}$ (defined for a simple complex Lie algebra $\mathfrak{g}$) is bounded, reaches its maximum $(\frac{248}{8^2} \approx 4)$ for type E_8:



(Lusztig 2013)

- Left cells of W (= Weyl group of $\mathfrak{g}$):

$$w \sim_L w' \quad \text{if} \quad I_w = I_{w'}$$

where I_w = primitive ideal in $U(\mathfrak{g})$ defined by the simple module with highest weight $-w(\rho) - \rho$. (Joseph 1970/80's)

Kazhdan–Lusztig (1979) + Lusztig (1983).

(W, S) arbitrary Coxeter system plus weights $\{p_s \mid s \in S\}$.

(p_s positive integer, $p_s = p_t$ if $s, t \in S$ conjugate in W.)

$\mathcal{H}$ = Iwahori–Hecke algebra over $\mathbb{C}(v)$:

$$\mathcal{H} = \bigoplus_{w \in W} \mathbb{C}(v)\, T_w, \qquad T_s^2 = T_1 + (v^{p_s} - v^{-p_s})\, T_s \ \text{ for } s \in S.$$

Let $\{C_w \mid w \in W\}$ be the new "canonical basis" of $\mathcal{H}$.

- $\rightsquigarrow$ Relations $\sim_L$, $\sim_R$, $\sim_{LR}$ on W (left, right and two-sided cells).
- $\rightsquigarrow$ A left $\mathcal{H}$-module $[\Gamma]$ and left W-module $[\Gamma]_1$ for each left cell Γ.
- $\rightsquigarrow$ Connections with geometry, combinatorics, representation theory.

In this talk: W finite – with special attention to $W = W(E_8)$.

- $|W| = 696{,}729{,}600$ with $\max\{l(w) \mid w \in W\} = 120$.
- $|\operatorname{Irr}(W)| = 112$ with $\max\{\dim E \mid E \in \operatorname{Irr}(W)\} = 7168$.
- Lusztig (1979) + Barbasch–Vogan (1983): There are 46 special representations $\leftrightarrow$ two-sided cells and 101,796 left/right cells.
- Lusztig (1986): There are 76 multiplicity vectors (m_E) such that $[\Gamma]_1 \cong \bigoplus_{E \in \operatorname{Irr}(W)} m_E\, E$, all explicitly known.
- Y. Chen (2000): Partition of W into left cells.
- Howlett (2003): Construction of W-graph representations for $\mathcal{H}$.
  $\rightsquigarrow$ explicit matrix models for $\operatorname{Irr}(\mathcal{H})$.
- G.–Müller (2009): Decomposition numbers of $\mathcal{H}$.
  $\rightsquigarrow$ applications to modular representations of Chevalley group $E_8(q)$.

What kind of questions are actually left to be solved?

Let C be a conjugacy class of involutions in W. Kottwitz (2000): Linear action of W on $\mathbb{C}$-vector space V_C with basis $\{e_w \mid w \in C\}$,

$$s.e_w = \begin{cases} -e_w & \text{if } sw = ws < w, \\ e_{sws} & \text{otherwise,} \end{cases} \qquad (s \in S).$$

(Formulation of Lusztig–Vogan, 2011.)

Kottwitz' Conjecture.

$$|C \cap \Gamma| = \dim \operatorname{Hom}_W(V_C, [\Gamma]_1) \quad \text{for every left cell } \Gamma \text{ of } W.$$

Conjecture now known to hold in general! Last case E_8: Halls (2013).

[Previous work: Kottwitz verified A_n; Casselman F_4, E_6; similar methods E_7; Bonnafé and G. (`hal-00698613`, `arXiv:1206.0443`): B_n, D_n.]

Need to be able to identify left cells Γ of the elements in a class C (as above), and corresponding modules $[\Gamma]_1$.

Let $G(q)$ = finite Chevalley group with Weyl group W, over $\mathbb{F}_q$.
For $w \in W$ let R_w be the virtual representation of $G(q)$ defined by Deligne and Lusztig. Unipotent representations:

$$\mathcal{U} := \{\rho \in \mathrm{Irr}_{\mathbb{C}}(G(q)) \mid (\rho : R_w) \neq 0 \text{ for some } w \in W\}.$$

Theorem (Lusztig 2002). Assume $\rho \in \mathcal{U}$ is cuspidal.
Then the set $\{w \in W \mid (\rho : R_w) \neq 0 \text{ and } l(w) = \text{minimum possible}\}$ is contained in a single conjugacy class of W, denoted C_ρ.

In type E_8 there exists ρ such that C_ρ consists of 4480 elements, all of order 6, length 40. To ρ is naturally attached a two-sided cell $\mathcal{F}_\rho$.

Observation: $\mathcal{F}_\rho$ contains exactly 4480 left cells.

Lusztig: *"This suggests that $C_\rho \subseteq \mathcal{F}_\rho$ and that any left cell in $\mathcal{F}_\rho$ contains a unique element of C_ρ"*. – Now verified !

Need to be able to reconstruct left cells of $w \in C_\rho$ efficiently.

Algorithmic challenge. (W, S) arbitrary, weights $\{p_s \mid s \in S\}$:

Given $w \in W$, find Γ such that $w \in \Gamma$, and $[\Gamma]_1 \cong \bigoplus_{E \in \mathrm{Irr}(W)} m_E E$.

Ways to approach left cells of W:

1. Original definitions: Kazhdan–Lusztig polynomials $P_{y,w}$ etc.
2. Induction from parabolic subgroups.

   (Will see that Vogan's generalised τ-invariant fits into this, too.)
3. Lusztig's **a**-invariant and leading coefficients ("limit" $v \to 0$).

What kind of answer do we expect for type E_8?

- Output file containing complete list of elements of W plus information on partition into cells. (Requires at least 5.6 GB.)
- Better: Efficient algorithms (applicable to any finite W) for specific tasks and further functionality (conjugacy classes, character tables of W, ...).

The easiest-to-compute invariant of a left cell:

> The function $w \mapsto \{s \in S \mid ws < w\}$ is constant on left cells.

Vogan's generalised τ-invariant is a far-reaching refinement of this.

Vogan I: *A generalized τ-invariant ...*, Math. Ann. (1979).

Vogan II: *Ordering of the primitive spectrum ...*, Math. Ann. (1980).

Will now give an explanation in terms of the concept of induction of cells – which works for arbitrary (W, S) and weights $\{p_s \mid s \in S\}$.

Consider parabolic subgroup $W_I \subseteq W$ where $I \subseteq S$. Let X_I be minimal left coset representatives $\rightsquigarrow$ projection map $\mathrm{pr}_I \colon W \to W_I$.

Theorem (Barbasch–Vogan 1983 and G. 2003). Let $y, w \in W$.

$$y \sim_L w \qquad \Rightarrow \qquad \mathrm{pr}_I(y) \sim_{L,I} \mathrm{pr}_I(w) \quad \text{(inside } W_I\text{)}.$$

Let $I \subseteq S$ and $\delta\colon W_I \to W_I$ be a bijection such that:

1. If Γ' is a left cell of W_I, then so is $\delta(\Gamma')$ and
2. the map $w \mapsto \delta(w)$ induces an isomorphism $[\Gamma'] \cong [\delta(\Gamma')]$.
3. The map $w \mapsto \delta(w)$ preserves the right cells in W_I.

Extend δ to $\delta^*\colon W \to W,\ xu \mapsto x\delta(u)$ (for $x \in X_I$, $u \in W_I$).

Example: Let $I = \{s,t\}$ where st has order 3; then $W_I \cong \mathfrak{S}_3$.

$$\text{Define} \qquad \delta : \begin{array}{cccc} \{1\} & \{s, ts\} & \{t, st\} & \{sts\} \\ \downarrow & \downarrow\ \downarrow & \downarrow\ \downarrow & \downarrow \\ \{1\} & \{st, t\} & \{ts, s\} & \{sts\} \end{array}$$

Then $\delta^*\colon W \to W$ is the $*$-operation of Vogan I/Kazhdan–Lusztig.

Note $p := p_s = p_t$; action of C_s and C_t on left cell module $[\{s, ts\}]$ given by

$$C_s \mapsto \begin{bmatrix} -v^p - v^{-p} & 1 \\ 0 & 0 \end{bmatrix}, \qquad C_t \mapsto \begin{bmatrix} 0 & 0 \\ 1 & -v^p - v^{-p} \end{bmatrix},$$

and we obtain exactly the same matrices for the action on $[\{st, t\}]$. So, (2) ok.

Let Δ be a collection of pairs (I, δ) satisfying the above conditions.

Definition (modelled on Vogan I). Let $y, w \in W$.

Define relation $y \approx_n w$ inductively for all $n \geqslant 0$:

- $y \approx_0 w$ if $\mathrm{pr}_I(y) \sim_{L,I} \mathrm{pr}_I(w)$ for all $(I, \delta) \in \Delta$.
- $y \approx_{n+1} w$ if $y \approx_n w$ and $\delta^*(y) \approx_n \delta^*(w)$ for all $(I, \delta) \in \Delta$.

Then y, w have the same "generalised τ^Δ-invariant" if $y \approx_n w$ $\forall n$.

Proposition (G. 2014). Two elements in the same left cell have the same generalised τ^Δ-invariant.

Let $\Delta =$ all pairs (I, δ) where $I = \{s, t\}$ with st of order 3 and δ as defined above. $\rightsquigarrow$ τ^Δ is Vogan's generalised τ-invariant.

Note: $p_s = p_t$ if $o(st) = 3$, but unequal weights on S now allowed!

Example: Let $I = \{s, t\}$ where st has order 4; then $|W_I| = 8$.

- If $p_s = p_t$, define $\delta : \begin{array}{cccc} \{1\} & \{s,\ ts, sts\} & \{t,\ st, tst\} & \{stst\} \\ \downarrow & \downarrow\ \downarrow\ \downarrow & \downarrow\ \downarrow\ \downarrow & \downarrow \\ \{1\} & \{sts, ts, s\} & \{tst, st, t\} & \{stst\} \end{array}$

  Then $\delta^*\colon W \to W$ is the map defined in Vogan II and also Lusztig (1985) ("method of strings").

- If $p_s < p_t$, define $\delta : \begin{array}{cccccc} \{1\} & \{s\} & \{t,\ st\} & \{ts, sts\} & \{tst\} & \{stst\} \\ \downarrow & \downarrow & \downarrow\ \downarrow & \downarrow\ \downarrow & \downarrow & \downarrow \\ \{1\} & \{s\} & \{ts, sts\} & \{t,\ st\} & \{tst\} & \{stst\} \end{array}$

  This yields a new "unequal weight $*$-operation" $\delta^*\colon W \to W$!

[Similar definitions possible for any $I = \{s, t\}$ where $o(st) \geqslant 3$.]

Conjecture/Question: Let $\Delta =$ all pairs (I, δ) where $|I| = 2$ and δ as above. Then $y, w \in W$ are in the same left cell iff y, w are in the same two-sided cell and y, w have the same generalised τ^Δ-invariant.

Specialisation $v \mapsto 1$ induces a bijection $\mathrm{Irr}(W) \leftrightarrow \mathrm{Irr}(\mathcal{H})$, $E \leftrightarrow E_v$.

$\mathcal{H}$ is a symmetric algebra $\rightsquigarrow$ orthogonality relations for $\mathrm{Irr}(\mathcal{H})$:

$$\sum_{w\in W} \mathrm{Trace}(T_w, E_v)^2 = (\dim E)\,\mathbf{c}_E, \qquad 0 \neq \mathbf{c}_E \in \mathbb{C}[v, v^{-1}].$$

Write $\mathbf{c}_E = f_E v^{-2\mathbf{a}_E} +$ higher powers of v, where $f_E > 0$, $\mathbf{a}_E \geqslant 0$.

$\rightsquigarrow$ Two invariants f_E, $\mathbf{a}_E$ of $E \in \mathrm{Irr}(W)$, explicitly known in all cases.

[Benson–Curtis, Iwahori, Tits' (1960/70); Lusztig (1979): Definition of f_E, $\mathbf{a}_E$.]

Proposition (G. 2002/2009).

$\exists$ basis of E_v such that $v^{\mathbf{a}_E} T_w \colon E_v \to E_v$ is represented by a matrix whose entries are rational functions in $\mathbb{C}(v)$ with no pole at $v = 0$.

$c_{w,E}^{ij} :=$ entry at (i,j) in matrix of $v^{\mathbf{a}_E} T_w$ evaluated at $v = 0$.

$c_{w,E} = \sum_i c_{w,E}^{ii}$ Lusztig's leading coefficient (1987).

Use leading coefficients to define a "limit $v \to 0$" algebra structure on $\tilde{\mathcal{J}} := \bigoplus_{w \in W} \mathbb{C} t_w$ which remembers information about cells.

For $x, y, z \in W$ set:

$$\tilde{\gamma}_{xyz} := \sum_{E \in \mathrm{Irr}(W)} f_E^{-1} \sum_{i,j,k} c_{x,E}^{ij}\, c_{y,E}^{jk}\, c_{z,E}^{ki}.$$

Then $\tilde{\mathcal{J}}$ is an associative algebra (with 1) with product

$$t_x \cdot t_y = \sum_{z \in W} \tilde{\gamma}_{xyz}\, t_{z^{-1}}.$$

The algebra $\tilde{\mathcal{J}}$ is semisimple and $\mathrm{Irr}(W) \leftrightarrow \mathrm{Irr}(\tilde{\mathcal{J}})$, $E \leftrightarrow E_{\tilde{\mathcal{J}}}$.

A matrix representation affording $E_{\tilde{\mathcal{J}}}$ is given by

$$\tilde{\rho}_E \colon \tilde{\mathcal{J}} \to M_d(\mathbb{C}), \qquad t_w \mapsto \left(c_{w,E}^{ij}\right)_{1 \leqslant i,j \leqslant d} \qquad (d = \dim E)$$

In the equal weight case, $\tilde{\mathcal{J}}$ is Lusztig's asymptotic algebra.
Some properties survive in the general weight case.

$\rightsquigarrow$ The numbers $c_{w,E}^{ij}$ and $c_{w,E}$ have some remarkable properties.

- Γ left cell $\Rightarrow$ multiplicity vector $(m_E)_{E\in\mathrm{Irr}(W)}$ given by

$$\sum_{w\in\Gamma} c_{w,E}\, c_{w,E'} = \begin{cases} f_E m_E & \text{if } E \cong E', \\ 0 & \text{otherwise.} \end{cases}$$

[Lusztig 1984 + G. 2014.]

- Γ left cell $\Rightarrow$ Γ contains at least one element of

$$\mathcal{D} := \Big\{ w \in W \mid \sum_{E\in\mathrm{Irr}(W)} f_E^{-1} c_{w,E} \neq 0 \Big\}.$$

[G. 2009; Lusztig 1987: "exactly one" in equal weight case, $\mathcal{D}$ is the set of distunguished (or Duflo) involutions in this case.]

- Let $y, w \in W$. Then y, w belong to the same left cell if

$$t_y \cdot t_{w^{-1}} \neq 0 \quad \Leftrightarrow \quad \sum_k c_{y,E}^{ik} c_{w^{-1},E}^{kj} \neq 0 \quad \text{for some } E, i, j.$$

[G. 2009; converse holds in equal weight case.]

$W = W(E_8)$: $|c_{w,E}| \leqslant 8$ for all w, E (Lusztig 1987).

[Explicit computation of all $c_{w,E}$ takes about 3 weeks on a computer; uses known "character table" of $\mathcal{H} = \mathcal{H}(E_8)$, G.–Michel 1997.]

What about $c^{ij}_{w,E}$? Lusztig (1981) + G. (2002):

- $\exists$ $\mathcal{H}$-invariant, non-degenerate symmetric $\beta_E : E_v \times E_v \to \mathbb{C}(v)$.
- If $\exists$ basis of E_v such that corresponding Gram matrix Ω_E of β_E has entries in $\mathbb{C}[v]$ and $\det(\Omega_E)|_{v=0} \neq 0$, then $c^{ij}_{w,E}$ defined.

Howlett (2003): Construction of W-graph representations for $\mathcal{H}(E_8)$.
G.–Müller (2009): Computation of Gram matrices Ω_E and verification of above property.

Hence, we can explicitly determine $c^{ij}_{w,E}$ and check relation $y \sim_L w$.
But: Can NOT do this for all 696,729,600 elements of W.

Recall: Family Δ of pairs $(I,\delta) \rightsquigarrow$ right $*$-operations $\delta^*\colon W \to W$.

("Right" because $\delta^*(xu) = x\delta(u)$ for $x \in X_I$, $u \in W_I$.)

Right $*$-orbit of a left cell Γ:

$\Gamma = \Gamma_0, \Gamma_1, \Gamma_2, \ldots$ where $\Gamma_{n+1} = \delta_n^*(\Gamma_n)$ for some $(I_n, \delta_n) \in \Delta$.

All $[\Gamma_n] \cong [\Gamma]$ have the same multiplicity vector $(m_E)_{E \in \mathrm{Irr}(W)}$.

Similarly, right $*$-orbit of a single element $w \in W$:

$w = w_0, w_1, w_2, \ldots$ where $w_{n+1} = \delta_n^*(w_n)$ for some $(I_n, \delta_n) \in \Delta$.

All w_n belong to the same right cell in W (Lusztig 2003).

Finally, the map $w \mapsto w^{-1}$ exchanges left and right cells.

If $R^*(w)$ is the right $*$-star orbit of $w \in W$, then the left $*$-star orbit $L^*(w) := \{y^{-1} \mid y \in R^*(w^{-1})\}$ is contained in a left cell of W.

Hence: Only need to store one left cell from each right $*$-orbit of left cells and, for each left cell Γ, only representatives of left $*$-orbits in Γ.

Proposition (G.–Halls 2014). $W = W(E_8)$.

- There are (only!) 106 right $*$-orbits of left cells of W and $[\Gamma]_1 \cong \bigoplus_E m_E E$ is explicitly known for these 106 cells.
- The union of the above 106 left cells consists of (only!) 678 left $*$-orbits (of which representatives are explicitly known).

So: Only need to store 678 elements ($+$ corresponding multiplicity vectors $(m_E)_{E\in \mathrm{Irr}(W)}$). Complete partition of W into left cells is reconstructed by repeated application of left/right $*$-operations.

Note: The 678 elements are difficult to compute, but once they are there, it is not so difficult to prove the above statements to be true.

Input: An element $w \in W$.

Output: The left cell Γ such that $w \in \Gamma$ and $[\Gamma]_1 \cong \bigoplus_E m_E E$.

Pre-processing: Apply left $*$-operations to produce the 106 left cells $\Gamma_1, \ldots, \Gamma_{106}$ out of the 678 representatives. (These contain 646,270 elements in total; the "unpacking" takes a couple of minutes.)

1. Compute the right $*$-orbit of w. This orbit meets a unique Γ_i. [Then w and Γ_i belong to the same two-sided cell, hence obtain unique "special" $E_0 \in \mathrm{Irr}(W)$ from multiplicity vector of $[\Gamma_i]_1$.]
2. Apply the right $*$-operations backwards to this Γ_i to obtain the full left cell Γ such that $w \in \Gamma$. Then $[\Gamma]_1 \cong [\Gamma_i]_1$.

Implemented and freely available in `PyCox` (written in `Python`)

(see LMS J. of Comput. and Math. **15**, 2012, and G.–Halls, `arXiv:1401.6804`).

The most difficult theoretical result involved in the algorithm:

There are 46 two-sided cells in $W(E_8)$.

(Lusztig's implication: $x \sim_{LR} y$ and $x \leqslant_L y \Rightarrow x \sim_L y$.)

Relies on positivity $(-1)^{l(x)+l(y)+l(z)} h_{xyz} \in \mathbb{N}_0[v, v^{-1}]$.

[Now purely algebraic proof due to Elias–Williamson (2012).]

Would like "$x \sim_{LR} y$ and $x \leqslant_L y \Rightarrow x \sim_L y$" in unequal weight case.

The most difficult computation involved:

Needed Gram matrices Ω_E for Howlett's W-graph representations.

Could then deduce that matrix of each $v^{\mathbf{a}_E} T_w \colon E_v \to E_v$ has entries given by rational functions in $\mathbb{C}(v)$ with no pole at $v = 0$.

(Note: Subsequently, matrices Ω_E are not required any more !)

Is there a theoretical argument to make this deduction ?