ON LANGLANDS' AUTOMORPHIC GALOIS GROUP and WEIL'S EXPLICIT FORMULAS

James Arthur.

F - number field

A: Galois group: $\Gamma_F = Gal(\bar{F}/F) = \varprojlim_E Gal(E/F)$, compact, tot. disconnected group.

B: Weil group: $W_F = W_{\bar{F}/F} = \varprojlim_E W_{E/F}$, locally compact group, with $W_F \twoheadrightarrow \Gamma_F$.

C: Langlands group: hypothetical loc. compact group (Langlands/Kottwitz), with $L_F \twoheadrightarrow W_F$, that would classify automorphic representations.

These groups have analogues for local completions F_v, with conj. classes of embeddings

$$\begin{tikzcd} L_{F_v} \arrow[r] \arrow[d, hook, "\text{(hypothetical)}"'] & W_{F_v} \arrow[r] \arrow[d, hook] & \Gamma_{F_v} \arrow[d, hook] \\ L_F \arrow[r, two heads] & W_F \arrow[r, two heads] & \Gamma_F \end{tikzcd} \qquad v \in \mathrm{val}(F),$$

where

$$L_{F_v} = \begin{cases} W_{F_v}, & \text{if } v \text{ is archimedean}, \\ W_{F_v} \times SU(2), & v \text{ nonarchimedean} \end{cases}$$

Suppose G/F reductive, $\pi \in \Pi_{aut}(G)$ (aut. rep of $G(\mathbb{A})$).

We get a family

$$c(\pi) = \{ c_v(\pi) = c(\pi_v) : v \notin S \}, \qquad \pi = \hat{\bigotimes_v} \pi_v,$$

of s.s. conj. classes in ${}^LG = \hat{G} \rtimes W_F$. Let

$$\mathcal{C}_{aut}(G) = \{ c(\pi) : \pi \in \Pi_{aut}(G) \}$$

be the set of equiv. classes of such families.

(where $c' \sim c$ if $c'_v = c_v$ for almost all v)

Principle of Functoriality (Conjecture of Langlands)

Suppose that $G, G'/F$ are quasisplit, and

$$\rho : {}^LG' \longrightarrow {}^LG$$

is an L-homo$^{\text{m}}$ (i.e. an analytic homo$^{\text{m}}$ over W_F). Then if $c' = \{c_v'\}$ lies in $\mathcal{C}_{aut}(G')$, its image

$$c = c(\rho') = \{\rho(c_v')\}$$

lies in $\mathcal{C}_{aut}(G)$.

If $r : {}^LG \longrightarrow GL(N, \mathbb{C})$, + $c \in \mathcal{C}_{aut}(G)$, define partial L-fn

$$L^S(s, c, r) = \prod_{v \notin S} \det(1 - r(c_v)\,|\varpi_v|^{-s})^{-1}.$$

Functoriality $\Longrightarrow$ $L^S(s, c, r)$ has an. cont. + f$^{\text{nal}}$ eq$^{\text{tn}}$.

Assume functoriality from now on.

Candidate for L_F : 2 ingredients

(I) Indexing set $\mathcal{C}_F$.

(II) For any $c \in \mathcal{C}_F$, an extn

$$1 \longrightarrow K_c \longrightarrow L_c \longrightarrow W_F \longrightarrow 1$$

of W_F by a compact, conn., simp. conn. gp K_c.

(I) If G/F is a q. split, simple, s.c gp, define $\mathcal{C}_{prim}(G)$ to be the set of $c \in \mathcal{C}_{aut}(G) \ni. \forall$ finite dim. rep. $r : {}^L G \longrightarrow GL(N,\mathbb{C})$,

$$-\operatorname{ord}_{s=1}(L^S(s,c,r)) = [r : 1_{{}^L G}] \quad \text{(mult. of trivial rep. in } r)$$

The indexing set is the set of isom classes of pairs

(I) $\qquad \mathcal{C}_F = \{(G, c) : c \in \mathcal{C}_{prim}(G)\}$

(II) Suppose $(G, c) \in \mathcal{C}_F$. Let

$$(*) \qquad 1 \longrightarrow Z \xrightarrow{\varepsilon} \tilde{G} \longrightarrow G_{ad} \longrightarrow 1$$

be a z-ext^n of G_{ad}. Then $G = \tilde{G}_{der} \subset \tilde{G}$.

e.g.: $G = SL(n)$, $G_{ad} = PGL(n)$, $\tilde{G} = GL(n)$, $Z = \mathbb{G}_m$.

Elem. hypothesis: $\exists$ $\tilde{c} \in \mathcal{C}_{aut}(\tilde{G})$ whose functorial image under the dual map ${}^L\tilde{G} \longrightarrow {}^LG$ is the given $c \in \mathcal{C}_{prim}(G)$.

Let

$$(\hat{*}) \qquad 1 \longrightarrow \hat{G}_{sc} \longrightarrow \hat{G} \xrightarrow{\hat{\varepsilon}} \hat{Z} \longrightarrow 1$$

be the dual exact seq. of $(*)$, + define

(i) $\hat{\varepsilon}(\hat{c}) \in \mathcal{C}_{aut}(Z)$ $\left(= c(\tilde{\chi})\right)$, $\tilde{\chi}$ the central char of any $\hat{\pi} \in \Pi_{aut}(\hat{G})$ with $c(\hat{\pi}) = \hat{c}$.

(ii) A dual 1-cocycle $z_c : W_F \longrightarrow \hat{Z}$ (Langlands' global classⁿ for torus Z)

(iii) A compact real form K_c of $\hat{G}_{sc}$, with normalizer $\hat{K}_c$ in $\hat{G}$.

(iv) $L_c = \{ g \rtimes w \in \hat{K}_c \rtimes W_F : \hat{\varepsilon}(g) = z_c(w) \}$.

This gives the extⁿ

(II) $1 \longrightarrow K_c \longrightarrow L_c \longrightarrow W_F \longrightarrow 1$.

Given the ingredients (I) + (II), we define

$$\boxed{L_F = \prod_{c \in \mathcal{C}_F} (L_c \longrightarrow W_F)}$$

– fibre product over W_F.

Local Hypothesis: Local Langlands corresp. holds: more precisely,

if $(G, c) \in \mathcal{C}_F \rightsquigarrow (\hat{G}, \tilde{c}) \rightsquigarrow \tilde{\pi} \in \Pi_{aut}(\hat{G})$ with $c(\tilde{\pi}) = \hat{c}$,

then $\forall v \in val(F)$, $\tilde{\pi}_v \in \Pi(\hat{G}_v) \rightsquigarrow \tilde{\Phi}_v : L_{F_v} \to {}^L\hat{G}_v$ and $\phi_v^1 \in H^1(L_{F_v}, \hat{\hat{G}}_v)$

such that $\hat{\Phi}_v$ and $\tilde{\Phi}_v^1$ depend only on $\hat{c}$ (+ not $\tilde{\pi}$).

Functoriality $\Longrightarrow$ Generalized Ramanujan $\Longrightarrow$ $im(\hat{\Phi}_v^1) \subset \hat{K}_c$

(Langlands: Bochner article SLN 170, p. 43–48)

Since $\hat{\varepsilon} \circ \hat{\Phi}_v^1$ projects to localization $z_{c,v} \in H^1(W_{F_v}, \hat{Z})$ of z_c,

$$\tilde{\Phi}_v^1(L_{F_v}) \subset \{ g_v \rtimes w_v \in \hat{K}_c \rtimes W_{F_v} : \hat{\varepsilon}(g_v) = z_{c,v}(w_v) \} \subset L_c,$$

so that $\phi_v^1 : L_{F_v} \longrightarrow L_c$. The fibre product L_F thus comes with embeddings $L_{F_v} \longrightarrow L_F$.

Conclusion: We have constructed an explicit loc. cpt group

$$L_F \twoheadrightarrow W_F,$$

with the conj. classes of local embeddings

$$\begin{tikzcd} L_{F_v} \arrow[r, two heads] \arrow[d, hook] & W_{F_v} \arrow[r] \arrow[d, hook] & \Gamma_{F_v} \arrow[d, hook] \\ L_F \arrow[r, two heads] & W_F \arrow[r, two heads] & \Gamma_F \end{tikzcd}, \qquad v \in \mathrm{val}(F).$$

<u>Problem</u>: Formulate a slightly broader hypothesis (<u>Functoriality Plus</u>) which

(i) would include functoriality + 2 hypotheses above

(ii) would be a necessary part of proof of functoriality proposed by Langlands in <u>Beyond Endoscopy</u>, and

(iii) would imply that L_F is <u>the</u> Langlands group — i.e. that it comes with a canonical bijection

$$\{\phi : L_F \longrightarrow GL(N, \mathbb{C}),\ \text{irred, unitary}\} \longleftrightarrow \{\pi \in \Pi_{cusp,\,unit}(GL(N))\}$$

compatible with localizations ϕ_v and π_v

Weil's explicit formula (See also Shin - Templier)

- Define $C_c^\infty(L_F)$ – space of $f^{\underline{ns}}$ that descend to $C_c^\infty(\kappa \backslash L_F)$, for normal sub$\underline{gp}$ $\kappa = \kappa_f$ of finite codim in L_F – i.e. $\ni$. $(\kappa \backslash L_F)$ is algebraic over $\mathbb{R}$.
- Define $L_F^1 = \ker(x \longrightarrow |x|)$ – compact gp, where $|x|$ is the abs. value of image of $x \in L_F$ in W_F. If

$$r : L_F \longrightarrow GL(N, \mathbb{C})$$

is an N-dim. rep. of L_F, so is

$$r_s(x) = r(x)|x|^s, \qquad s \in \mathbb{C}.$$

- Given r, we can form

$$L(s,r) = \prod_v L(s, r_v) = L_S(s,r) L^S(s,r),$$

where r is "unramified" outside $S = S_r \supset S_\infty$. Since we are assuming functoriality, $L(s,r)$ has an. cont. + f$\underline{nal}$ eq$\underline{tn}$.

Define $\mathcal{Z}_F(r) = \{\rho \in \mathbb{C} : L(\rho, r) = 0\}$.

For any $f \in C_c^\infty(L_F)$, the sum

$$\sum_{\rho \in \mathcal{Z}_F(r)} tr(r_\rho(f))$$

depends only on rest^n of r to L_F^1. We can then form the "spectral sum"

$$\sum_{r \in \Pi(L_F^1)} \sum_{\rho \in \mathcal{Z}_F(r)} tr(r_\rho(f)), \qquad f \in C_c^\infty(L_F),$$

where $\Pi(L_F^1)$ is set of equiv. classes of irred. (finite dim) rep^s of L_F^1.

· Given $f \in C_c^\infty(\gamma_f \backslash L_F) \subset C_c^\infty(L_F)$, $\exists$ finite set $S = S_f \supset S_\infty$ of $\mathrm{val}(F) \ni$ if $v \notin S$, γ_f contains the inertia sub gp $I_{F_v} \times SU(2) \subset L_{F_v} \subset L_F$, and hence gives a Frobenius conj. class $c = c_v$ in $\gamma \backslash L_F$.

We can then form the "orbital integral"

$$\int_{L_{F,c} \backslash L_F} f(x^{-1} \gamma x)\, dx, \qquad L_{F,c} = \mathrm{Cent}(c, L_F),$$

and the "geometric sum"

$$\sum_{c \in C_F^S} \sum_{n \in \mathbb{N}} \mathrm{vol}(C_{F,c}^S \backslash L_{F,c}) \int_{L_{F,c} \backslash L_F} f(x^{-1} c^n x)\, dx,$$

where

$$C_F^S = \{ c = c_v : v \notin S = S_f \}, \quad \underline{\text{and}} \quad C_{F,c}^S = \{ c^k : k \in \mathbb{Z} \} \subset L_{F,c}$$

THEOREM: Suppose that f is symmetric, in the sense that $f(x)$ equals $f^\vee(x) = f(x^{-1})|x|$. Then the "geometric side"

$$- \sum_{c \in C_F^S} \sum_{n \in \mathbb{N}} \mathrm{vol}(C^S_{F,c} \backslash L_{F,c}) \int_{L_{F,c} \backslash L_F} f(x^{-1} c^n x)\, dx$$

$$+ \sum_{r \in \Pi(L_F^1)} \frac{1}{2\pi i} \int_{Re(z) = \frac{1}{2}} \frac{L_S'(z,r)}{L_S(z,r)} \, tr(r_z(f))$$

equals the "spectral side"

$$\frac{1}{2} \sum_{r \in \Pi(L_F^1)} \sum_{\rho \in \mathfrak{Z}_F(r)} tr(r_\rho(f)) - \int_{L_F} f(x)\, dx .$$