# Galois and $\theta$ Cohomology

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# Erratum

David and Lois know each other from High School

So M is locally a product of $SL(2,\mathbb{R})$ and a compact torus. We will divide the argument more or less arbitrarily into several parts, for the convenience of the reader who is reading during the commercials only.

*Step I.* $X(\gamma)$ *and* $\mathrm{Ind}_P^G(\delta \otimes \nu \otimes 1)$ *have the same set of lambda-*

One more thing.

## Galois Cohomology

$F$ local $\quad \Gamma = \mathrm{Gal}(\overline{F}/F)$ $G = G(\overline{F})$ defined over $F$

$G(F) = G(\overline{F})^{\Gamma}$

$H^i(\Gamma, G) =$ Galois cohomology $\quad$ (group cohomology)

$i = 0, 1$ if $G$ is not abelian

Example: $G(F) = GL(n, F): \quad H^1(\Gamma, G) = 1$

$(GL(1, F) = F^*$ : Hilbert's Theorem 90)

Example: $G = SO(V)$:

$H^1(\Gamma, G) = \{$ non-degenerate quadratic forms of same dimension and discriminant as $V\}$

Example: $G = Sp(2n, F)$

$H^1(\Gamma, G) = \{$ non-degenerate symplectic forms, dim. $2n\} = 1$

# Rational Forms

Basic Fact:

$$\{\text{rational (inner) forms of } G\} \longleftrightarrow H^1(\Gamma, G_{\text{ad}})$$

(Inner: $\sigma'\sigma^{-1}$ is inner)

NB: (for the experts): equality of rational forms is by the action of $G$, not $\text{Aut}(G)$ (Borel: $\text{Image}(H^1(\Gamma, G_{\text{ad}}) \to H^1(\Gamma, \text{Aut}(G_{\text{ad}})))$

Theorem (Kneser): $F$ p-adic, $G$ simply connected $\Rightarrow H^1(\Gamma, G) = 1$

Not true over $\mathbb{R}$ ... $G(\mathbb{R}) = SU(2)$, $H^1(\Gamma, G) = \mathbb{Z}/2\mathbb{Z})$

Problem: Calculate $H^1(\Gamma, G)$ $G$ simply connected, defined over $\mathbb{R}$

This fact plays a role in statements about the trace formula, functoriality, packets. . .

## Real case

$F = \mathbb{R}$, $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle$

$H^0(\Gamma, G) = G(\mathbb{R})$

$H^1(\Gamma, G) = \{g \in G \mid g\sigma(g) = 1\}/[g \to xg\sigma(x^{-1})]$

Write $H^i_\sigma(\Gamma, G)$

Digression: $H$ =torus, $\widehat{H}^i(\Gamma, H)$ Tate cohomology

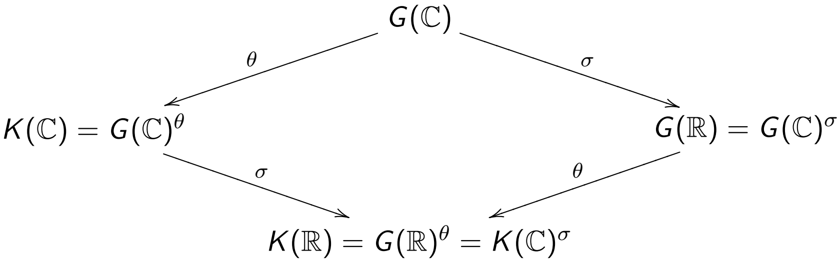
$\widehat{H}^0(\Gamma, H) = H(\mathbb{R})/H(\mathbb{R})^0$

Question: Is it possible (and a good idea?) to define $\widehat{H}^i(\Gamma, G)$ ($i = 0, 1$) in such a way that $\widehat{H}^0(\Gamma, G) = G(\mathbb{R})/G(\mathbb{R})^0$?

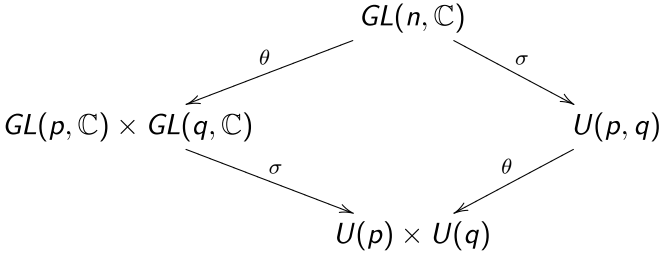
## Real Forms: $\sigma$ and $\theta$ pictures

Cartan: classify real forms by their Cartan involution: a real form is determined by its maximal compact subgroup $K(\mathbb{R})$ - in fact by $K(\mathbb{C}) = G(\mathbb{C})^{\theta}$

$\theta$ is a holomorphic involution.



## Real Forms: $\sigma$ and $\theta$ pictures




**Theorem:** (Cartan)

$$\{\sigma \mid \sigma \text{ antiholomorphic}\}/G \longleftrightarrow \{\theta \mid \sigma \text{ holomorphic}\}/G$$

$$\sigma \longrightarrow \theta = \sigma\sigma_c$$

$$\sigma = \theta\sigma_c \longleftarrow \theta$$

## Real Forms: $\sigma$ and $\theta$ pictures

$\sigma$, $\theta$ pictures are deeply embedded in representation theory

$\sigma$ : $G(\mathbb{R})$ acting on a Hilbert space

$\theta$ : $(\mathfrak{g}, K)$ modules $\mathfrak{g}, K$ both complex

Matsuki duality (later): $X = G(\mathbb{C})/B(\mathbb{C})$

$$G(\mathbb{R})\backslash X \longleftrightarrow K(\mathbb{C})\backslash X$$

Kostant-Sekiguchi correspondence (nilpotent $G(\mathbb{R}), K(\mathbb{C})$ orbits)

# Real Forms: $\theta$ cohomology

$\theta$ holomorphic:

Definition: $H^i_\theta(\mathbb{Z}_2, G)$: group cohomology with $\mathbb{Z}_2$ acting by $\theta$

$$H^0_\theta(\mathbb{Z}_2, G) = K$$

(remember $K = K(\mathbb{C})$)

$$H^1_\theta(\mathbb{Z}_2, G) = \{g \mid g\theta(g) = 1\}/[g \to xg\theta(x^{-1})]$$

Basic Point: $H^1_\theta(\mathbb{Z}_2, G)$ is much easier to compute than $H^1_\sigma(\Gamma, G)$

Example: $\theta = 1$:

$$H^1_\theta(\mathbb{Z}_2, G) = \{g \mid g^2 = 1\}/G = \{h \in H \mid h^2 = 1\}/W$$

$G(\mathbb{R})$ compact
(Serre): $H^1(\Gamma, G) = H^1(\Gamma, G(\mathbb{R})) = \{h \in H(\mathbb{R}) \mid h^2 = 1\}/W$

## Real Forms: Example

$\theta = 1$, $H^1(\mathbb{Z}_2, G) = H_2/W$:

Exercise:

$G = E_8$, $R$=root lattice, $|R/2R| = 256$

$$|H^1_\theta(\mathbb{Z}_2, G)| = |(R/2R)/W| = 3 \quad (1 + 120 + 135 = 256)$$

## Galois and $\theta$ cohomology

$H^1_\theta(\mathbb{Z}_2, G) \quad H^1_\sigma(\Gamma, G)$

Cartan's Theorem can be stated: $\sigma \leftrightarrow \theta \Rightarrow$

$$H^1_\sigma(\Gamma, G_{\mathrm{ad}}) \simeq H^1_\theta(\mathbb{Z}_2, G_{\mathrm{ad}})$$

Question: drop the adjoint condition?

Theorem: $G$ connected reductive,

$\sigma$ antiholomorphic, $\theta$ holomorphic
$\sigma \leftrightarrow \theta$ (in the sense of Cartan; i.e. defining the same real form)
There is a canonical isomorphism:

$$\boxed{H^1_\sigma(\Gamma, G) \simeq H^1_\theta(\mathbb{Z}_2, G)}$$

## Sketch of proof

(1) $H$ torus: $1 \to H_2 \to H \stackrel{z^2}{\to} H \to 1$

$|\Gamma = 2| \Rightarrow$

$H^1_\sigma(\Gamma, H) \simeq H^1_\sigma(\Gamma, H_2)$

$H^1_\theta(\mathbb{Z}_2, H) \simeq H^1_\theta(\mathbb{Z}_2, H_2)$

and $\theta|_{H_2} = \sigma|_{H_2}$

$$H^1_\sigma(\Gamma, H) \simeq H^1_\sigma(\Gamma, H_2) = H^1_\theta(\mathbb{Z}_2, H_2) \simeq H^1_\theta(\mathbb{Z}_2, H)$$

(2) $H_f$ a fundamental (most compact) Cartan subgroup;

$$H^1_\sigma(\Gamma, H_f) \twoheadrightarrow H^1_\sigma(\Gamma, G)$$

(easy: every semisimple elliptic element is conjugate to an element of $H_f$)

## Sketch of proof (continued)

(3) $W_i(H)$ = Weyl group of imaginary roots,

$$H^1_\sigma(\sigma, H)/W_i(H) \hookrightarrow H^1_\sigma(\Gamma, G)$$

This is non-trivial but standard:

it comes down to $(G/P)(F) = G(F)/P(F))$ (Borel-Tits) and there is only one conjugacy class of compact Cartan subgroups (very special to $\mathbb{R}$)

Equivalently: over $\mathbb{R}$ stable conjugacy of Cartan subgroups is equivalent to ordinary conjugacy (Shelstad) (false in the p-adic case)

$$\boxed{H^1_\sigma(\Gamma, G) \simeq H^1_\sigma(\Gamma, H_f)/W_i}$$

Theorem (Borovoi): $H^1_\sigma(\Gamma, G) \simeq H^1_\sigma(\Gamma, H_f)/W^\sigma$

Exactly same argument holds for $\theta$-cohomology:

$$\boxed{H^1_\theta(\mathbb{Z}_2, G) \simeq H^1_\theta(\mathbb{Z}_2, H_f)/W_i}$$

## Applications

### Two versions of the rational Weyl group

$W_\sigma = \mathrm{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R})$

$W_\theta = \mathrm{Norm}_{K(\mathbb{C})}(H(\mathbb{C}))/H(\mathbb{C}) \cap K(\mathbb{C})$

Theorem (well known, see Warner): $W_\sigma \simeq W_\theta$

# Applications

proof:

$$1 \to H \to N \to W \to 1$$

$$1 \to H^\sigma \to N^\sigma \to W^\sigma \to H^1_\sigma(\Gamma, H)$$

$$\begin{array}{ccccccc}
1 & \longrightarrow & W_\sigma = N^\sigma/H^\sigma & \longrightarrow & W^\sigma & \longrightarrow & H^1_\sigma(\Gamma, H) \\
 & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \simeq} \\
1 & \longrightarrow & W_\theta = N^\theta/H^\theta & \longrightarrow & W^\theta & \longrightarrow & H^1_\theta(\Gamma, H)
\end{array}$$

# Applications

## Matsuki Correspondence of Cartan subgroups

Theorem (Matsuki): There is a canonical bijection

$$\{\sigma\text{-stable } H\}/G(\mathbb{R}) \leftrightarrow \{\theta\text{-stable } H\}/K$$

Proof in quasisplit case:

$$LHS = H^1_\sigma(\Gamma, W) \simeq H^1_\theta(\mathbb{Z}_2, W) = RHS$$

(general: $H^1_\sigma(\Gamma, N) \simeq H^1_\theta(\mathbb{Z}_2, N)\dots$)

# Applications: Strong real forms

For simplicity: assume equal rank inner class

**Definition** (ABV) A **strong real form** of $G$ is $G$-conjugacy class of $x \in G$ satisfying $x^2 \in Z(G)$.

$\{$strong real forms$\} \twoheadrightarrow \{$real forms$\}$ (bijection if $G$ is adjoint)

$$x \to \theta_x = \mathrm{int}(x) \text{ (conjugation by } x)$$

Pure Real forms: $x^2 = 1$

$$\begin{array}{ccc}
\{\text{strong real forms}\} & \twoheadrightarrow & H^1(\Gamma, G_{\mathrm{ad}}) = \{\text{real forms}\} \\
\uparrow & & \uparrow \\
H^1(\Gamma, G) = \text{pure real forms} & \twoheadrightarrow & [\text{image}]
\end{array}$$

Problem:

1) Give a cohomological definition of strong real forms

2) Define "strong rational forms" of p-adic groups
(Kaletha): $H^1(u \to W, Z \to Z)$ =strong real forms in real case

# Applications: Strong real forms

Strong Real Forms:

$$x \to \mathrm{inv}(x) = x^2 \in Z^\Gamma$$

Real forms:

$$\mathrm{inv} : H^1(\Gamma, G_{\mathrm{ad}}) \to H^2(\Gamma, Z) = \widehat{H}^0(\Gamma, Z) = Z^\Gamma/(1+\sigma)Z$$

Theorem: Given $\sigma \to \mathrm{inv}(\sigma) \in Z^\Gamma/(1+\sigma)Z \to$ (choose) $z \in Z^\Gamma$

$$\boxed{H^1(\Gamma, G) \leftrightarrow \{\text{strong real forms of type } z\}}$$

$\to$: classical Galois cohomology interpretation of strong real forms

$\leftarrow$: compute $H^1(\Gamma, G)$(the right hand side is easy)

## Applications: Strong real forms

Corollary:

$$\{\text{strong real forms}\} \longleftrightarrow \bigcup_{z \in S} H^1_{\sigma_z}(\Gamma, G)$$

$S = Z^\Gamma/(1+\sigma)Z$

$S \ni z \to \sigma_z \ (\sigma_z \leftrightarrow \theta_x \to x^2 = z)$

## Application: Computing $H^1(\Gamma, G)$

Compute {strong real forms of type $z$}

(equal rank case):

$$H^1(\Gamma, G) \simeq \{g \in G \mid g^2 = z\}/G = \{h \in H \mid h^2 = z\}/W$$

($z$ depends on the real form)

Example:

$G = Sp(2n, \mathbb{R}) \quad x = \mathrm{diag}(i, \dots, i, -i, \dots, -i) \quad z = -I$:

$$H^1_\sigma(\Gamma, G) = \{g \mid g^2 = -I\}/G = \{\mathrm{diag}(\pm i, \dots, \pm i)\}/W = 1$$

Example:

$G = Spin(p, q)$

$SO(p, q)$: $\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$ (classifying quadratic forms)

$Spin(p, q)$: $\lfloor \frac{p+q}{4} \rfloor + \delta(p, q) \quad \delta(p, q) = 0, 1, 2, 3$

## 9.1 Classical groups

| Group | $\lvert H^1(\Gamma, G)\rvert$ | |
|---|---|---|
| $SL(n, \mathbb{R}), GL(n, \mathbb{R})$ | 1 | |
| $SU(p, q)$ | $\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$ | Hermitian forms of rank $p + q$ and discriminant $(-1)^q$ |
| $SL(n, \mathbb{H})$ | 2 | $\mathbb{R}^*/\mathrm{Nrd}_{\mathbb{H}/\mathbb{R}}(\mathbb{H}^*)$ |
| $Sp(2n, \mathbb{R})$ | 1 | real symplectic forms of rank $2n$ |
| $Sp(p, q)$ | $p + q + 1$ | quaternionic Hermitian forms of rank $p + q$ |
| $SO(p, q)$ | $\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$ | real symmetric bilinear forms of rank $n$ and discriminant $(-1)^q$ |
| $SO^*(2n)$ | 2 | |

Here $\mathbb{H}$ is the quaternions, and $Nrd$ is the reduced norm map from $\mathbb{H}^*$ to $\mathbb{R}^*$ (see [12, Lemma 2.9]). For more information on Galois cohomology of classical groups see [13], [12, Sections 2.3 and 6.6] and [9, Chapter VII].

## 9.2 Simply connected groups

The only simply connected groups with classical root system, which are not in the table in Section 9.1 are $Spin(p, q)$ and $Spin^*(2n)$.

Define $\delta(p, q)$ by the following table, depending on $p, q \pmod 4$.

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 3 | 2 | 2 | 2 |
| 1 | 2 | 1 | 1 | 0 |
| 2 | 2 | 1 | 0 | 0 |
| 3 | 2 | 0 | 0 | 0 |

| Group | $\lvert H^1(\Gamma, G)\rvert$ |
|---|---|
| $Spin(p, q)$ | $\lfloor \frac{p+q}{4} \rfloor + \delta(p, q)$ |
| $Spin^*(2n)$ | 2 |

| Simply connected exceptional groups | | | | | |
|---|---|---|---|---|---|
| inner class | group | $K$ | real rank | name | $\lvert H^1(\Gamma, G)\rvert$ |
| compact | $E_6$ | $A_5A_1$ | 4 | quasisplit' quaternionic | 3 |
| | $E_6$ | $D_5T$ | 2 | Hermitian | 3 |
| | $E_6$ | $E_6$ | 0 | compact | 3 |
| split | $E_6$ | $C_4$ | 6 | split | 2 |
| | $E_6$ | $F_4$ | 2 | quasicompact | 2 |
| compact | $E_7$ | $A_7$ | 7 | split | 2 |
| | $E_7$ | $D_6A_1$ | 4 | quaternionic | 4 |
| | $E_7$ | $E_6T$ | 3 | Hermitian | 2 |
| | $E_7$ | $E_7$ | 0 | compact | 4 |
| compact | $E_8$ | $D_8$ | 8 | split | 3 |
| | $E_8$ | $E_7A_1$ | 4 | quaternionic | 3 |
| | $E_8$ | $E_8$ | 0 | compact | 3 |
| compact | $F_4$ | $C_3A_1$ | 4 | split | 3 |
| | $F_4$ | $B_4$ | 1 | | 3 |
| | $F_4$ | $F_4$ | 0 | compact | 3 |
| compact | $G_2$ | $A_1A_1$ | 2 | split | 2 |
| | $G_2$ | $G_2$ | 0 | compact | 2 |

### 9.3 Adjoint groups

If $G$ is adjoint $|H^1(\Gamma, G)|$ is the number of real forms in the given inner class, which is well known. We also include the component group, which is useful in connection with Corollary 8.3.

One technical point arises in the case of $PSO^*(2n)$. If $n$ is even there are two real forms which are related by an outer, but not an inner, automorphism. See Remark 2.2.

| Adjoint classical groups | | |
|---|---|---|
| Group | $\lvert\pi_0(G(\mathbb{R}))\rvert$ | $\lvert H^1(\Gamma, G)\rvert$ |
| $PSL(n, \mathbb{R})$ | $\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$ | $\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$ |
| $PSL(n, \mathbb{H})$ | 1 | 2 |
| $PSU(p, q)$ | $\begin{cases} 2 & p = q \\ 1 & \text{otherwise} \end{cases}$ | $\lfloor \frac{p+q}{2} \rfloor + 1$ |
| $PSO(p, q)$ | $\begin{cases} 1 & pq = 0 \\ 1 & p, q \text{ odd and } p \neq q \\ 4 & p = q \text{ even} \\ 2 & \text{otherwise} \end{cases}$ | $\begin{cases} \lfloor \frac{p+q+2}{4} \rfloor & p, q \text{ odd} \\ \frac{p+q}{4} + 3 & p, q \text{ even}, p + q = 0 \pmod 4 \\ \frac{p+q-2}{4} + 2 & p, q \text{ even}, p + q = 2 \pmod 4 \\ \frac{p+q+1}{2} & p, q \text{ opposite parity} \end{cases}$ |
| $PSO^*(2n)$ | $\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$ | $\begin{cases} \frac{n}{2} + 3 & n \text{ even} \\ \frac{n-1}{2} + 2 & n \text{ odd} \end{cases}$ |
| $PSp(2n, \mathbb{R})$ | 2 | $\lfloor \frac{n}{2} \rfloor + 2$ |
| $PSp(p, q)$ | $\begin{cases} 2 & p = q \\ 1 & \textit{else} \end{cases}$ | $\lfloor \frac{p+q}{2} \rfloor + 2$ |

The groups $E_8, F_4$ and $G_2$ are both simply connected and adjoint. Furthermore in type $E_6$ the center of the simply connected group $G_{\text{sc}}$ has order 3, and it follows that $H^1(\Gamma, G_{\text{ad}}) = H^1(\Gamma, G_{\text{sc}})$ in these cases. So the only groups not covered by the table in Section 9.2 are adjoint groups of type $E_7$.

| Adjoint exceptional groups | | | | | | |
|---|---|---|---|---|---|---|
| inner class | group | $K$ | real rank | name | $\pi_0(G(\mathbb{R}))$ | $\lvert H^1(G)\rvert$ |
| compact | $E_7$ | $A_7$ | 7 | split | 2 | 4 |
| | $E_7$ | $D_6A_1$ | 4 | quaternionic | 1 | 4 |
| | $E_7$ | $E_6T$ | 3 | Hermitian | 2 | 4 |
| | $E_7$ | $E_7$ | 0 | compact | 1 | 4 |