Geometry of PDEs

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ICM
July 11 2022

Geometry & analysis

Optimal geometries & evolution of shapes are governed by PDEs.

Geometric invariance makes the PDE canonical.

It means:

1. The same types of equations appear over and over again across many diverse areas.
2. The equations also describe phenomena seemingly unrelated to geometry.

Geometry & analysis

Often the geometry unlocks the structure
leading to fundamental tools in PDE.

Understanding the equations requires insight into both:

Analysis & geometry

plus the interplay between the two.

We will see examples of this.

We will emphasize a few big ideas & themes,

suppressing many other interesting aspects & results.

Hopefully, this will give a taste of a very large & active area,
focusing on joint work with Bill Minicozzi.

Interplay in 3 examples

Interplay between analysis & geometry in 3 examples:

- In optimal regularity in PDEs.
- In stability of solutions.
- In geometry of diffeomorphism group.

Existence vs regularity

In PDEs existence is often shown weakly
(like distribution or viscosity).

Challenge:
Proving regularity.

Nonlinear PDE:
Weak solutions might not be smooth.

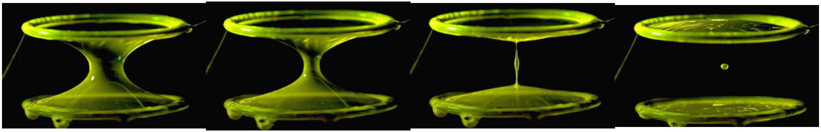
How smooth?
How big is the singular set?

Stability of solutions

In Dynamics & PDEs a central question is:

Which solutions are stable?

– which are not?



The stable are the ones that physically happen

– the unstable ones are not seen in nature.

Geometry of ∞-dim'l diffeomorphism group

Geometric properties are universal
– independent of coordinates.

Yet objects look very different in different coordinates.

How do we recognize geometric objects
when no canonical coordinates exist?

Understand the geometry of the diffeomorphism group.

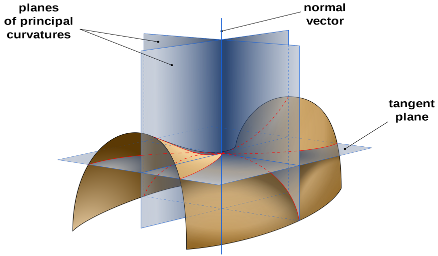
Surface tension gives a PDE that describes evolution of shape.

Mean curvature **H** is the force from surface tension.

Effect of surface tension

Surface tension causes water to form roughly spherical droplets.

Mean curvature **H** of a surface is sum of principal curvatures.



For a level set of a function u,

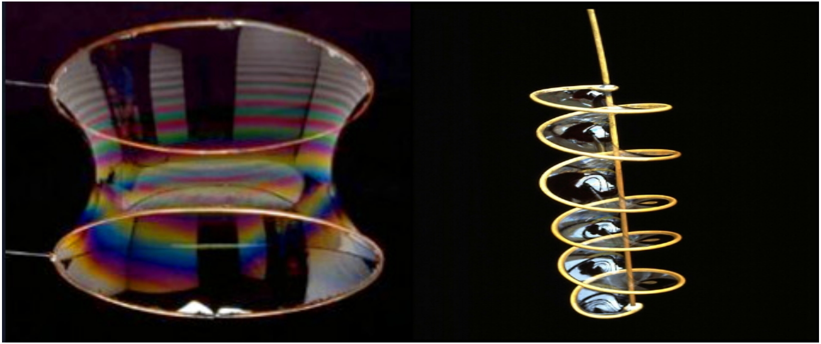
$$\vec{\mathbf{n}} = \frac{\nabla u}{|\nabla u|}$$

is a unit normal and

$$\mathbf{H} = \operatorname{div} \frac{\nabla u}{|\nabla u|} \,.$$

Equilibrium: Minimal surfaces

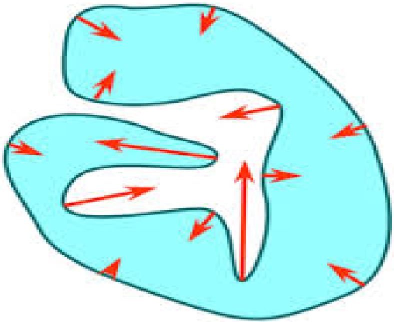
If surface tension is only force acting,
then equilibrium occurs when $\mathbf{H} = 0$. These are minimal surfaces.



Minimal surfaces have a long history
beginning with Euler and Lagrange.

A surface evolves in time
by each point moving normal $\vec{\mathbf{n}}$ to the surface with speed $\mathbf{H}$:

$$x_t = -\mathbf{H}\,\vec{\mathbf{n}}\,.$$



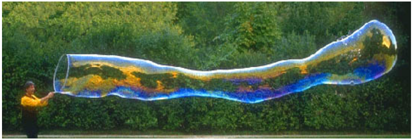
Convex points move inward
– concave points move out.

Mathematics of surface tension

Mean curvature flow is a nonlinear heat equation.

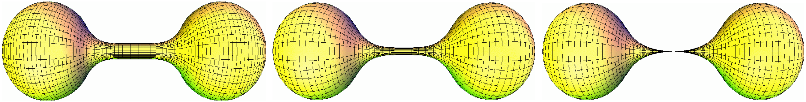
It is the (negative) gradient flow for area
on the infinite dimensional space of surfaces:

The flow makes the area shrink as fast as possible.

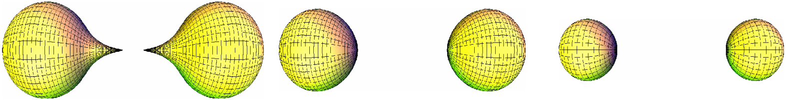


Mean curvature flow goes back more than 100 years
in mathematics & material science.

Example: Evolution of Grayson's dumbbell



Initial dumbbell, shrinking neck, & neck pinch singularity.



Cusps retract & each piece becomes round.

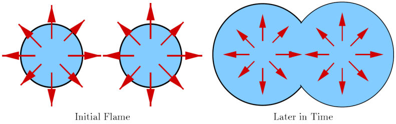
An $\mathbf{S}^1$ of singularities

Marriage ring:

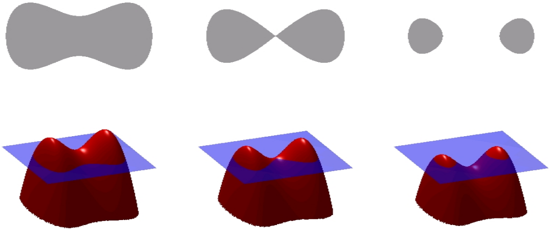
A torus of revolution remains a surface of revolution under the flow, becoming extinct along a round $\mathbf{S}^1$.

Level Set Method from applied mathematics

Tracking moving front = Level Set Method.



Idea: Represent evolving front as level sets of a function:

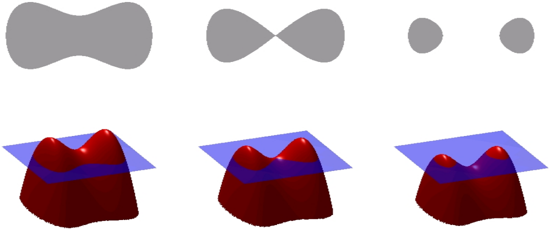


Which PDE does the function satisfy?

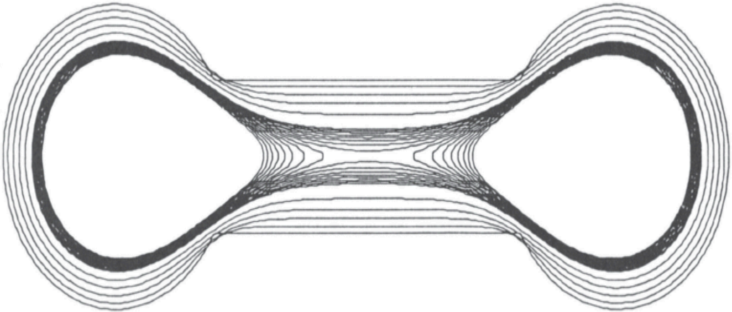
Level Set Method

The Level Set Method allows for:

Singularities & topological changes.



Monotone front: Flow that moves inward.



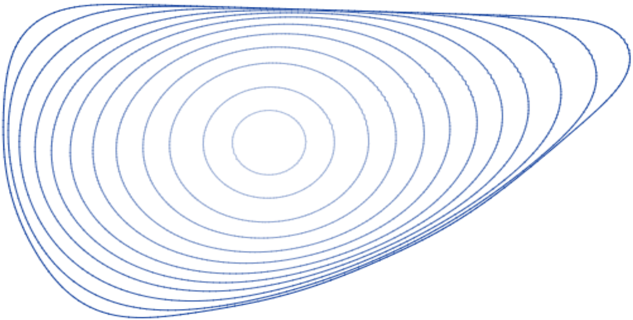
Arrival time function u: $u^{-1}(t)$ are the fronts.

$u(x)$ the time when the front arrives at x.

u defined on domain that initial front bounds.

Arrival time equation

Evolving curves are level sets of u:



$$-1 = |\nabla u| \operatorname{div} \frac{\nabla u}{|\nabla u|} .$$

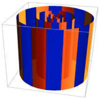
Degenerate elliptic equation.

Spheres & cylinders

Arrival time functions on $\mathbf{R}^3$:

For spheres becoming extinct at the origin at time 0:

$$-\frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$$



For cylinders becoming extinct in the line $x_2 = x_3 = 0$ at time 0:

$$-\left(x_2^2 + x_3^2\right) .$$

Arrival time equation

Central in applied math, going back to Osher-Sethian.

Arises also in game theory.

Fits into a larger family of natural PDEs.

Existence & regularity

Evans-Spruck, Chen-Giga-Goto:
Viscosity solutions exist & are Lipschitz.

Fundamental question:
How smooth are solutions?

Examples of Ilmanen: NOT C^2 in general;
cf. Huisken, Kohn-Serfaty, Sesum.

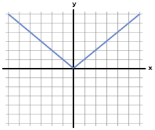
A continuous function $u : \mathbf{R}^n \to \mathbf{R}$ has $\Delta\, u \geq 0$ at $x = 0$ in the viscosity sense if

$\exists$ a smooth barrier function v with

- $v(0) = u(0)$,
- $u \geq v$,
- $\Delta\, v \geq 0$ at $x = 0$.

Maximum principle holds for such u.

E.g., $u(x) = |x|$ has $\Delta\, |x| \geq 0$ at $x = 0$.



Optimal regularity & geometry

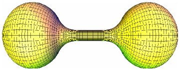
Thm (**CM**): The arrival time is twice differentiable everywhere.

Not always C^2!

Being C^2 has geometric meaning:

Thm (**CM**): The arrival time is C^2 iff the entire evolving front becomes singular at the same time & then extinct.

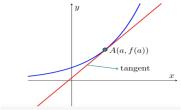
C^2 only if it is like a marriage ring or sphere,

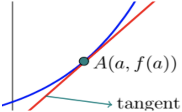


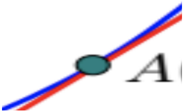
dumbbell NOT C^2.

Differentiability & uniqueness

A function is differentiable if it looks like
the **same** linear function on all sufficiently small scales.



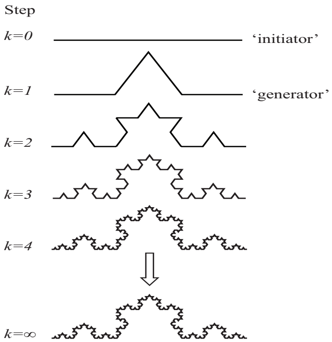


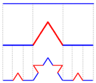


Derivatives as unique limit of rescalings.

Koch curve – uniqueness fails

Koch curve is a fractal; it is an iterative limit of broken lines.





When each broken line is almost flat:

On each scale the curve looks roughly like a line.

Yet under larger magnifications looks like a different line.

Rescaling of arrival time functions

The flow & u are both smooth away from critical points, i.e., away from points where $\nabla u = 0$.

If 0 is critical point define rescalings

$$v_{\lambda}(x) = \lambda^{-2}\, u(\lambda\, x)\,.$$

v_{λ} satisfies same equation.

Homogeneous quadratic polynomials are preserved.

Two examples: cylinders & spheres:

- Both have quadratic polynomials as arrival time.
- For both v_{λ} is independent of λ.

Rescaling & regularity

Twice differentiable means:

There is a 2nd order Taylor expansion.

Thus must show that $v_\lambda(x)$ has a limit as $\lambda \to 0$.

A priori – no reason to expect any limit!

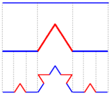
– Even for a subsequence.

Huisken-Ilmanen-White get geometric blowups
– but depend on choice of subsequence.

For the flow:
Blowups $=$ dilation-invariant solutions called shrinkers.
Most blowups are non-compact.

Might be like Koch?

For Koch: One sequence of rescalings gives one blowup,
another gives a different blowup.



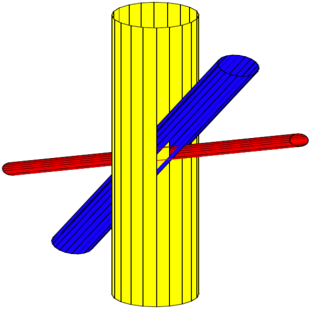
For monotone flows, possible limits are classified:
Spheres & cylinders,

by Huisken-Sinestrari, White,
Haslhofer-Kleiner, Andrews, CM.

This classification relies on
a matrix maximum principle by Hamilton.

Are limits unique?

Or, do different subsequences give different limits?



Uniqueness

Thm (**CM**): Uniqueness of blowups for all monotone flows.

Questions of uniqueness
have long been recognized in geometry
as fundamental.

Fundamental work of Allard-Almgren, Simon
on uniqueness questions for minimal varieties.

Uniqueness & twice differentiabilty

Thm (**CM**): Uniqueness of blowups implies:

u looks like the same quadratic polynomial at all small scales.

This gives the 2nd order Taylor expansion
& twice differentiability.

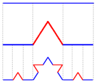
Common in nonlinear PDE:

Solutions can have singularities.

Marriage ring:
The singular set can be a curve (1-dimensional).

White (via dimension reduction):
1-dim'l, but measure $= \infty$?

Koch curve has infinite length & non-unique blowups.



Key to bound the singular set: Uniqueness.

Thm (**CM**): Finite 1-dim'l measure.

Similar results hold in all dim.

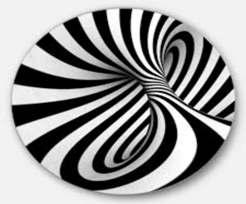
Dynamics

Uniqueness can be understood dynamically for rescaled flow.

Rescaled flow $=$ magnifying continuously along the flow.

Uniqueness $\Leftrightarrow$ solution of rescaled flow with a limit point has a unique limit.

This contrasts with wandering points in dynamics:



Gaussian surface area & monotonicity

Rescaled flow is a gradient flow.

Gaussian surface area of n-dim'l submanifold Σ:

$$F(\Sigma) = (4\,\pi)^{-\frac{n}{4}} \int_{\Sigma} e^{-\frac{|x|^2}{4}} \, .$$

Huisken:
F monotone $\downarrow$ for rescaled flow around origin.

Singularities at origin
$=$ critical points of F
$=$ equilibrium for rescaled flow.

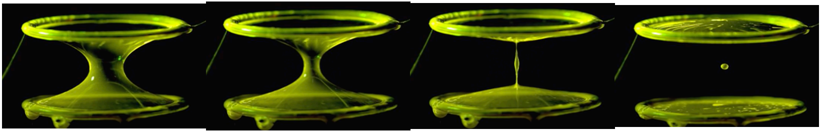
Lyapunov function

Entropy =
supremum of Gaussian surfaces areas
over all dilations $t_0 > 0$ & translations x_0

$$\lambda(\Sigma) = \sup_{t_0 > 0, x_0} F(t_0 \, \Sigma + x_0) \, .$$

Entropy is a Lyapunov function for mean curvature & all rescaled mean curvature flow.

The catenoid is in equilibrium for surface tension,

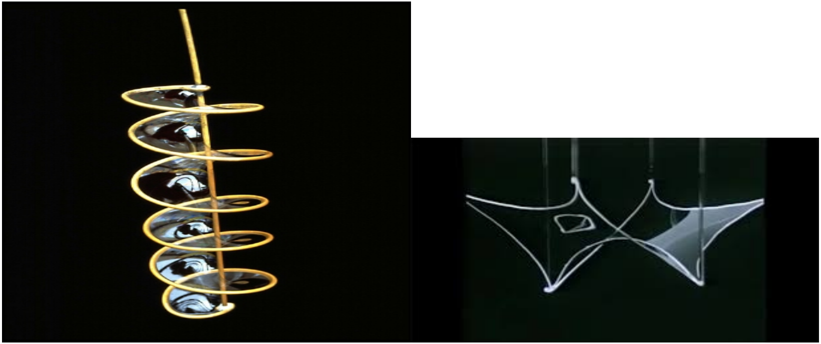


perturbing it ever so slightly changes it completely.

$$\Downarrow$$

It is unstable & will never occur in nature.

Stable structures



For the flow (not necessarily monotone):

What singularities are stable?
Which cannot be perturbed away?

Stability

In dynamical systems stability near an equilibrium was investigated by Lyapunov.

If the flow starting out near an equilibrium stays near the equilibrium forever, then the equilibrium is Lyapunov stable.

Generic singularities

Generic singularities

=

singularities that cannot be perturbed away.

Thm (**CM**):

Only generic singularities:

Shrinking spheres & cylinders – in all dim.

Recognizing singularities

As one approaches a singularity & magnifies
at any time, one only sees part of the singularity.

Closer to the singularity one sees more,
yet always only a finite part.

Is a finite part enough
to recognize the singularity?

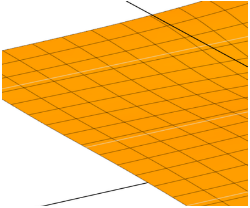
Can one recognize the whole from a finite part?

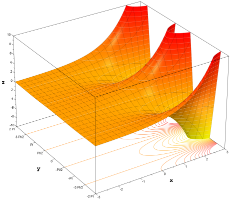
Not in PDEs!

Closeness on a piece does NOT typically fix a solution.

A nontrivial harmonic function on $\mathbf{R}^n$ can be arbitrarily small on a large set.

The real part of $(x, y) \to e^{x+iy}$:





When $x << 0$. When $0 << x$.

Can one recognize the whole from a finite part?

Not in geometry!

Ricci flat gravitational instantons contain

arbitrarily large, almost Euclidean regions.

Photo: Tamas Hausel.

Shrinker principle

Contrast:
Strong rigidity for most important singularities
requires just the compact piece
& just that one knows roughly what it looks like.

This is the *shrinker principle:*
Uniqueness radiates outwards.

Originally discovered in mean curvature flow:

Thm (**C-Ilmanen-M**):
Generic singularities are strongly rigid;
a compact piece determines the whole space.

For most PDEs it is unreasonable to expect that a finite part determines the entire space.

The shrinker principle is an example of how the geometry unlocks the structure of the PDE.

Shrinker principle in Ricci flow

Shrinker principle holds for other equations.

Ricci flow is a system of PDEs (introduced by Hamilton) that describes the evolution of a metric on a manifold.

Thm (**CM**): The *shrinker principle* also holds for Ricci flow.

Special case shown independently by Li-Wang, using work of Brendle and Kotschwar.

A major issue in Ricci flow: Gauge group.

To recognize a metric one needs to look at it the “right” way.
In the wrong coordinate system it would be unrecognizable:

Metrics that could even be the same
look very different in different coordinates.

Need to understand ∞-dim'l group of diffeomorphisms
(gauge group) on non-compact shrinkers.

To find “right” gauge: Solve a “canonical” non-linear PDE that finds a diffeo, Φ.

Right gauge is orthogonal to the action of ∞-dim’l group of diffeo’s.

Show optimal bounds for the displacement:

$$x \to \text{dist}(x, \Phi(x)) \,,$$

that measures

“how different the right gauge is from the given one.”

Concluding remarks

We have discussed several canonical equations,
where many key issues come up.

Seen interplay between analysis & geometry.

In particular, how:

- Analysis helps understand geometry.
- Geometry unlocks analysis,
  – Optimal regularity.
  – Stability of solutions.
  – Geometry of diffeomorphism group.