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Summary, February 26 and March 1

I plan to explain the proof of the Main Theorem.

We need to be precise about isomorphic representations. An isomorphism of representations $G \xrightarrow{\rho} V$ an $G \xrightarrow{\rho'} V'$ is an isomorphism of vector spaces $V' \xrightarrow{T} V$ that is compatible with the two operations of G: $T(gv') = gT(v')$ for all g in G and all v' in V'. This relation can also be written as

$$T\rho'_g = \rho_g T$$

for all g. I like to display this relation as a diagram:

(0.0.1)
$$\begin{array}{ccc} V' & \xrightarrow{T} & V \\ {\scriptstyle \rho'_g}\big\downarrow & & \big\downarrow{\scriptstyle \rho_g} \\ V' & \xrightarrow{T} & V \end{array}$$

A linear transformation $V' \xrightarrow{T} V$, not necessarily an isomorphism, with this property is called an *inveriant transformation*. An isomorphism of representations is an invariant isomorphism of vector spaces.

Invariant transformations provide a method to compare two representations. However, as the next lemmas explain, it is very hard for a linear transformation to be invariant. For instance, if ρ' is a trivial representation, i.e., V' has dimension one and $\rho'_g = 1$ for all g, then $T(v') = T(gv')$. In this case, $T(gv') = gT(v')$ only if $T(v')$ is an invariant vector.

Lemma 1. If T is an invariant linear transformation, then the kernel of T is an invariant subspace of V' and the image of T is an invariant subspace of V.

proof. Let K be the kernel of T. To show that K is invariant, we must show that if x is in K, then gx is in K for all g in G. Since T is invariant, $T(gx) = gT(x) = g0 = 0$. Yes, gx is in K.

Let W be the image of T. We must show that if y is in W, then gy is in W for all g. Since y is in W, $y = T(v')$ for some v'. Then $gy = gT(v') = T(gv')$ So gy is $T(something)$, and is in the image W. □

Schur's Lemma. (i) Let $G \xrightarrow{\rho} V$ and $G \xrightarrow{\rho'} V'$ be irreducible representations. A nonzero invariant transformation $V' \xrightarrow{T} V$ is an isomorphism. If ρ and ρ' are not isomorphic, the only invariant transformation $V' \xrightarrow{T} V$ is zero.
(ii) Let $G \xrightarrow{\rho} V$ be an invariant representation, and let T be an invariant transformation $V \to V$. Then T is multiplication by a scalar: $T(v) = cv$ for some $c \in \mathbb{C}$.

proof. (i) Let K and W be the kernel and image of T, respectively. Since K is an invariant subspace of V', either $K = 0$ or $K = V'$, and if $T \neq 0$, then $K = 0$. Similarly, since W is an invariant subspace of V, $W = 0$ or $W = V$, and if $T \neq 0$, then $W = V$. So, if $T \neq 0$, then $k = 0$, $W = V$, and T is an invariant isomorphism.

(ii) Suppose given a nonzero invariant transformation $V \xrightarrow{T} V$. Since ρ is irreducible, T, **(i)** tells us that T is an isomorphism, an invertible linear operator on V. We choose an eigenvector v for this operator. Say $T(v) = cv$, and we inspact the linear operator $S = T - cI$ defined by $S(w) = T(w) - cw$. This operator is invariant, and v is in its kernel. Since its kernel is not zero, S is the zero operaator, and $T = cI$. □

Now: Though Schur's Lemma shows that it is very hard for a linear transformation to be invariant, we can use the averaging process to produce an invariant transformation from an arbitrary transformation. The averaging process is as follows: Let ρ and ρ' be given representations, and let $V' \xrightarrow{T} V$ be an arblitrary linear transformation. Then

$$\widetilde{T} = \frac{1}{|G|}\sum_g \rho_g^{-1} T \rho_g$$

is invariant. To show this, we must show that for every h in G, $\widetilde{T}\rho'_h = \rho_h\widetilde{T}$, or $\widetilde{T} = \rho_h^{-1}\widetilde{T}\rho_h$. We expand:

$$\rho_h^{-1}\widetilde{T}\rho_h = \frac{1}{|G|}\sum_g \rho_h^{-1}\rho_g^{-1}T\rho_g\rho'_h = \frac{1}{|G|}\sum_g \rho_{gh}^{-1}T\rho'_{gh}$$

As g runs through the group, so does $g_1 = gh$. Therefore $\rho_h^{-1}\widetilde{T}\rho_h = \frac{1}{|G|}\sum_g \rho_{gh}^{-1}T\rho_{gh} = \frac{1}{|G|}\sum_{g_1}\rho_{g_1}^{-1}T\rho_{g_1} = \widetilde{T}$.

Corollary 1. If ρ and ρ' are non-isomorphic irreducible representations, the transformation $\widetilde{T}$ produced by averaging from a linear transformation $V' \xrightarrow{T} V$ is zero.

Let ρ and ρ' be representations of G. The set L of linear transformations $V' \to V$ is a vector space. (When bases for the spaces V' and V are given, L becomes isomorphic to the space of $m \times n$ matrices, $m = \dim V$ and $n = \dim V'$.) The averaging operator on the space L will be denoted by Φ. When $V' \xrightarrow{T} V$ is an element of L,

$$\Phi(T) = \frac{1}{|G|}\sum_g \rho_g^{-1}T\rho_g$$

Lemma 2. Let A and B be $n \times n$ and $m \times m$ matrices. Let F be the linear operator on the space of $m \times n$ matrices defined by $F(M) = AMB$. The trace of F is the product $\tau\, A\, \tau\, B$.

You can prove this lemma, or refer to the text 10.8.1. □

Lemma 3. Let χ be the character of a representation ρ: $\chi(g) = \tau\,\rho_g$. Then $\tau\,\rho_g^{-1} = \overline{\chi(g)}$.

proof. By definition, $\chi(g) = \tau\,\rho_g$ is the sum of the eigenvalues of ρ_g. Since G is a finite group, the element g has finite order. Therefore all of its eigenvalues λ have finite order. They lie on the unit circle in the complex plane. If λ is on the unit circle, then $\lambda^{-1} = \overline{\lambda}$. Then the eigenvaues of $\rho_{g^{-1}}$ are the complex conjugates of the eigenvalues of ρ_g, and $\tau\,\rho_{g^{-1}} = \overline{\tau\,\rho_g} = \overline{\chi(g)}$. □

Proposition. *(trace of the averaging operator)* Let χ and χ' be the characters of ρ and ρ'. The trace of Φ is $<\chi, \chi'>$.

proof. Since trace is a linear operation, Lemma 2 shows that

$$\operatorname{trace}\Phi = \frac{1}{|G|}\sum_g \operatorname{trace}\rho_{g^{-1}}\operatorname{trace}\rho'_g = \frac{1}{|G|}\sum_g \overline{\chi(g)}\chi'(g) = <\chi, \chi'>$$

□

Corollary 2. If ρ and ρ' are non-isomorphic irreducible representations, then $<\chi, \chi'> = 0$.

proof. When ρ and ρ' are irreducible and not isomorphic, Corollary 1 asserts that Φ is the zero operator. Its trace, which is $<\chi, \chi'>$, is zero. □

Lemma. Let $\widetilde{L}$ and K denote the image and kernel of Φ, respectively.
(i) L is the direct sum $\widetilde{L} \oplus K$.
(ii) The trace of Φ is equal to $\dim \widetilde{L}$.

proof. **(i)** This follows from the fact that $\Phi^2 = \Phi$: If T is invariant, then $\widetilde{T} = T$. So $\Phi^2(T) = \Phi(\widetilde{T}) = \widetilde{T}$. (The factor $\frac{1}{|G|}$, which is often irrelevant, is important here.)

(ii) The trace of Φ is the sum of the traces of its restrictions to $\widetilde{L}$ and K. The restriction to K is the zero operator, and the rstriction to $\widetilde{L}$ is the identity. Its trace is equal to $\dim \widetilde{L}$. □

Corollary 3. If χ is the character of an irreducible representation, then $<\chi, \chi> = 1$.

proof. Schur's Lemma tells us that the invariant transformations $V \to V$ are multiplication by scalars. This means that the space $\widetilde{L}$ of invariant transformations has dimension one. By part **(ii)** of the previous lemma, $\operatorname{trace}\Phi = 1$. The proposition above tells us that $\operatorname{trace}\Phi = <\chi, \chi>$. □

To complete the proof that the irreducible characters form an orthonormal basis for the space of class functions (functions that are constant on conjugacy classes), we must still show that the irreducible characters span that space. We do this by showing that a class function that is orthogonal to every character is zero.

Let $\mathcal{H}$ be the space of class functions, and let $\mathcal{C}$ be the subspace spanned by the characters. The hermitian form on $\mathcal{H}$ is the one used for characters. If u and v are class functions, then $< u, v >= \frac{1}{|G|}\sum_g \overline{u(g)}v(g)$.

Let φ be a class function that is orthogonal to every character χ: $< \varphi, \chi >= \frac{1}{|G|}\sum_g \overline{f(g)}\chi(g)$. Say that χ is the character of the representation $G \xrightarrow{\rho} GL(V)$.

We form a linear operator T on V, a combination of the operators ρ_g, defining $T = \frac{1}{|G|}\sum_g \overline{f(g)}\rho_g$.

Lemma. For any representation ρ,
(i) the operator T is invariant. In fact
(ii) T is the zero operator.

proof. **(i)** We must show that for any h in G, $\rho_h^{-1}T\rho_h = T$. We substitute into the definition, remembering that $\overline{f(g)}$ is a scalar. Let $g'' = h^{-1}gh$. As g runs through the group, so does g'', and since f is a class function, $f(g) = f(g'')$. Then

$$\rho_{h^{-1}}T\rho_h = \frac{1}{|G|}\sum_g \overline{f(g)}\rho_{h^{-1}}\rho_g\rho_h = \frac{1}{|G|}\sum_g \overline{f(g)}\rho_{g''} = \frac{1}{|G|}\sum_{g''} \overline{f(g'')}\rho_{g''} = T$$

(ii) We may assume that ρ is irreducible.

Let $c(g)$ be the scalar coefficient $\frac{1}{|G|}\overline{f(g)}$ of ρ_g in T. Since trace is linear, the trace of T is

$$\operatorname{trace} T = \sum_g c(g)\operatorname{trace}\rho_g = \frac{1}{|G|}\sum_g \overline{f(g)}\chi(g) =< f, \chi >$$

Since f is orthogonal to χ, $\operatorname{trace} T = 0$. Since T is invariant, Schur's Lemma tells us that $T = cI$ for some scalar c. Then because the trace is zero, $T = 0$. $\square$

We apply the lemma to the regular representaton rho^{reg}. Recall that this is the permutation representation of G operating on itself by multiplication. The lemma tells us that $\frac{1}{|G|}\overline{f(g)}\rho_g^{reg} = 0$. We notice that the operators ρ_g^{reg} are independent. If we order G with 1 as the first element of the list, the matrix R_g^{reg} of $\rho^{reg})g$ will have a single nonzero entry, a 1, in the first column at the place where g appears in the list. Thus the first columns are independent.

This being so, the relation $\sum_g \overline{f(g)}\rho_g^{reg} = 0$ tells us that the coefficients $\overline{f(g)}$ are zero for all g, and therefore that $f = 0$, as we wanted to show.