Summaries, February22 and 24

We determine some character tables.

I. Let G be the tetrahedral group of symmetries of a regular tetrahedron, which is also the alternating group A_4. Its order is 12. The conjugacy classes were probably discussed in 18.701. Unfortunately, we can't show the process easily, so we display the table. Init$x$ denotes a rotation by angle $2\pi/3$ about a vertex and y denotes rotation by π about an edge. There are four conjugacy classes, so four irreducible characters. Let their dimensions be $d_1, ..., d_4$. The formula $|G| = d_1^2 + \cdots + d_r^2$ shows that $d_i = 1, 1, 1, 3$. This determines $\chi_i(1)$ for $i = 1, 2, 3, 4$.

The character χ is the one corresponding to the operation A_4 by permutations of four indices. So $\chi(y)$ is the number of indices fixed by y, which is a product of two disjoint transpositions. That number is zero. The character χ is a sum of irreducible chacters, including th trivial character χ_1. This determines chi_4.

Finally, let ρ be the one-dimensional representation of G whose character is χ_2. Then $\chi_2(x)$ is the unique eigenvalue of the one-dimensional operator ρ_x. Since $x^3 = 1$, it is also true that $(\rho_x)^3 = 1$ and that $\chi_2(x)^3 = 1$. There are three possibilities: $\chi_2(x) = 1, \omega$ or ω^2, with $\omega = e^{2\pi i}/3$. Moreover, $\chi_2(x^2)$ is the unique eigenvalue of ρ_x^2, which is the square of $\chi_2(x)$. The three possibilities for χ_2 are the first three rows of the table.

chartableT (0.0.1)

	(1)	(3)	(4)	(4)
	1	y	x	x^2
$\chi_1:$	1	1	1	1
$\chi_2:$	1	1	ω	ω^2
$\chi_3:$	1	1	ω^2	ω
$\chi_4:$	3	-1	0	0
$\chi:$	4	0	1	1

II. Let G be the dihedral group D_5 of symmetries of a regular pentagon. Let x denote rotation by $2\pi/5$, and let y be one of the reflection symmetries. The elements of G, grouped into conjugacy classes, are $1, \{x, x^4\}, \{x^2, x^3\}, \{y, xy, x^2y, x^3y, x^4y\}$ The dimensions of the irreducible characters are $1, 1, 2, 2$.

The character χ in the bottom row is the permutation character in which G operates on the vertices of the pentagon. It is the sum $\chi_1 + \chi_3 + \chi_4$.

The character χ_3 is the one that corresponds to the operation of G on the plane, in which x is rotation by $2\pi/5$ and y is a reflection. The value of the character on x is $\alpha = 2\cos 2\pi/5 = \zeta + \zeta^{-1}$, with $\zeta = e^{2\pi i/5}$, and $\beta = 2\cos 4\pi/5 = \zeta^2 + \zeta^3$.

The character table is

(0.0.2) chartableD5

	(1)	(2)	(2)	(5)
	1	x	x^2	y
$\chi_1:$	1	1	1	1
$\chi_2:$	1	1	1	-1
$\chi_3:$	2	α	β	0
$\chi_4:$	2	β	α	0
$\chi:$	5	0	0	1

III. Let G be an arbitrary finite group. The permutation reprsentation in which G operates on itself by left multiplication is called the *regular representation*, and its character χ_{reg} is the *regular character*. Then $\chi_{reg}(g)$

is the number of elements of G fixed by left multiplication by g. That number is zero unless g is the identity, in which case it is the order of the group: $\chi_{reg}(1) = |G|$. If χ_i is an irreducible character of G,

$$\langle \chi_{reg}, \chi_i \rangle = \frac{1}{|G|} \sum_g \chi_{reg}(g)\chi_i(g) = \frac{1}{|G|}\chi_{reg}(1)\chi_i(1) + 0 + \cdots + 0 = \frac{1}{|G|}|G|\chi_i(1) = \dim \chi_i$$

Let d_i be the dimension of χ_i The projection formula $\chi_{reg} = \sum_i \langle \chi_{reg}, \chi_i \rangle \chi_i$ shows that $\chi_{reg} = \sum_i d_i \chi_i$. Therefore $|G| = \dim \chi_{reg} = \sum d_i \dim \chi_i = \sum d_i^2$. This proves one part of the Main Theorem !

February 24.

Summing over the group. Let ρ be a representation of G on V. Because G is finite, one can sum over the group. This is a way to produce something that is invariant.

The simplest examples start with a subspace W of V. The sum $U = \sum_g gW$ of the subspaces gW is invariant, and so is the intersection $T = \bigcap_g gW$: For any group element h, $U = hU$ and $T = hT$.

The reason that these subspaces are invariant subspaces is that, as g runs over the group, so does $g' = hg$, though in a different order.

For example, let $G = S_3$, and let $h = y$. As g runs through the group in the order $1, x, x^2, y, xy, x^2y$, $g' = hg$ runs through G in the order $y, x^2y, xy, 1, x^2, x$.

Therefore $\sum_g hg = sum_g g' = \sum_{g'} g' = \sum_g g$, and

$$hU = \sum_g g'W = \sum_g gW = U$$

Similarly,

$$hT = \bigcap_g hgW = \bigcap_g g'W = \bigcap_g gW = T$$

The next example is averaging an element v of the vector space V. The averaging operation is

$$\widetilde{v} = \frac{1}{|G|} \sum_g gv$$

If h is a group element, then $h\widetilde{v} = \frac{1}{|G|} \sum_g hgv$. We put $g' = hg$: $h\widetilde{v} = \frac{1}{|G|} \sum_g g'v$. As g runs over the group, so does g', in a different order. Therefore the sum $\sum_g g'$ is equal to $\sum_{g'} g'$, and $h\widetilde{v} = \frac{1}{|G|} \sum_g gv = \widetilde{v}$.

However, it may very well happen that $\widetilde{v}$ is the zero vector. So this averaging process isn't always interesting.

The factor $\frac{1}{|G|}$ that appears isn't important. It is there so that, if v happens to be inveriant itself, then $\widetilde{v} = v$

Next, let $[\,,\,]$ be a positive definite hermitian form on V. The form is called *invariant* if $[v, w] = [gv, gw]$ for all g. If the form is invariant, the operators ρ_g will be *unitary*.

The averaging process can be used to produce an invariant form from an arbitrary form.

We start with an arbitrary positive definite hermitian form $\{\,,\,\}$ on V. For instance, we could choose a basis for V and carry the standard hermitian form on $\mathbb{C}^n$ over using the basis. We define a new form $[\,,\,]$ by

$$[v, w] = \frac{1}{|G|} \{gv, gw\}$$

This form is positive definite and invariant. To prove that it is invariant, we show that $[v, w] = [hv, hw]$ for all h in G:

$$[hv, hw] = \frac{1}{|G|} \sum_g \{ghv, ghw\} = \frac{1}{|G|} \sum_g \{g'v, g'w\}$$

As g runs over the group, so does $g'' = gh$, though in a different order. Therefore

$$[hv, hw] = \frac{1}{|G|} \sum_{g''} \{g''v, g''w\} = [v, w]$$

V. Proof of Maschke's Theorem The theorem asserts that every representation is a direct sum of irreducible representations. To prove it, we start with a representation ρ on a space V. If there is no proper invariant subspace, then ρ is irreducible. If there is a proper invariant subspace W, we look for a complementary subspace W' such that V is the direct sum $W \oplus W'$. If W' exists, we can apply induction on the dimension to conclude that the restrictions of ρ to W and W' are direct sums of irreducible repreentations, and then ρ will be a sum of irreducibles too.

We choose an invariant positive definite form $[\,,\,]$ on V, so that $[v, w] = [\rho_g v, \rho_g w]$ for all g in G and all v, w in V. I hope you have earned that this formula shows that ρ_g are *unitary operators*. (See Proposition 8.6.3

of the text. A unitary operator preserves orthogonality. Therefore, if W is invariant, $W = \rho_g W$, and if W' is the orthogonal space $W^\perp$, then $\rho_g W'$ will be the orthogonal space to $\rho_g W = W$. So $W' = \rho_g W'$, i.e., W' is invariant.

Character table for the icosahedral group.

Let G be the icosahedral group of rotational symmetries of a regular icosahedron or dodecahedron. It is isomorphic to the alternating group A_5. The conjugacy classes were described in 18.701, I hope. They can be identified by the angles of rotation. or by the type of permutation of five indices. I've displayed two permutation representations below. The first is the operation of A_5 on five indices. The second is the operation of the icosahedral group on the six pairs of opposite faces of a dodecahedron. For example, a rotation x by $2\pi/5$ fixes the axis of rotation, i.e., one pair of opposite faces. So $\chi_{f.pr}(x) = 1$. Rotation by $2\pi/3$ fixes no pair of opposite faces. Looking at a picture of the dodecahedron, I can't see the face pairs fixed by a rotation by π about an edge, so the number (2) is in parentheses. It can be seen to be the only possible value by orthogonality with the trivial representation.

(0.0.3) charsA5

	(1)	(15)	(20)	(12)	(12)	
	0	π	$2\pi,$	$32\pi/5$	$4/5$	$(angle)$
	(.)	(..)(..)	(...)	(.....)	(.....)	$(perm)$
$\chi_{perm}:$	5	1	2	0	0	
$\chi_{f.pr.}:$	6	(2)	0	1	1	

Subtracting the trivial character from χ_{perm} and from $\chi_{f.pr.}$ gives two of the irreducible representations. One also has the representation of 3-space by rotations. Its character can be computed easily. Remember that the trace of rotation by θ on 3-space is $1 + 2\cos\theta$, the 1 resulting from the fact that the rotation fixes its axis.

With this information, the character table is computed easily. It is in the text.