Summary-February19.tex

As before, $GL(V)$ is the group of invertible linear operators on the vector space V. A representation of G on V is a homomorphism

$$G \xrightarrow{\rho} GL(V)$$

Its character χ is the map $G \xrightarrow{\chi} \mathbb{C}$ (not a homomorphism) defined by

$$\chi(g) = trace \rho_g$$

The dimension of the space V is also called the dimension of the representation ρ and also called the dimension of the character χ.

An invariant subspace W of V is a subspace such that $gW \subset W$, or $\rho_g W \subset W$. Since ρ_g is invertible, the inclusions will be equalities.

An irreducible representation is one such that V contains no proper invariant subspace.

The character is constant on conjugacy classes. The *character table* for $G = S_3$, which is below, is obtained by listing each conjugacy class just once.

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	(1)	(2)	(3)
	1	x	y
χ_{triv} :	1	1	1
χ_{sign} :	1	1	-1
χ_{stand} :	2	-1	0
χ_{perm} :	3	0	1

In this table, the top row lists the orders of the conjugacy classes. The second row identifes the conjugacy classes by an element in each class. The next three rows list the irreducible characters, and the bottom row lists the character of the representation of S_3 by 3×3 permutation matrices. (See below.)

The first column of a character table, which lists $\chi(1)$ always consists of the dimensions of the representations, because $\chi(1)$ is the trace of the identity matrix.

To compute $\langle \chi_{triv}, \chi_{stand} \rangle$, we must count each product with multiplicity as in the top row:

$$\langle \chi_{triv}, \chi_{stand} \rangle = (1) \cdot 1 \cdot 2 + (2) \cdot 1 \cdot (-1) + (3) \cdot 1 \cdot 0 = 2 + (-2) + 0 = 0$$

So χ_{triv} and χ_{stand} are orthogonal, as the Main Theorem asserts.

Main Theorem The irreducible characters form an orthonormal basis for the class functions (functions on conjugacy classes, or functions on G that are constant on conjugacy classes).

Let's go back to the character χ_{perm}. **Permutation representations.** If a group G operates on a set S, its element act as permutations, and the corresponding permutation matrices define a matrix representation $G \xrightarrow{R} GL_n$ of the group. There are two things to note:

• The character $\chi(g)$ is the sum of the diagonal entries of the matrix R_g. A diagonal entry 1 corresponds to an element of S that is fixed by g. So $\chi(g)$ is the number of fixed elements.

• The vector $(1, 1, ..., 1)^t$ is fixed by every permutation. It is a fixed vector that spans a one-dimensional subspace on which G operates trivially. Every permutation representation contains the trivial representation as a summand.

In the character χ_{perm}, $G = S_3$ is operating by permutations of three elements: x permutes the three cyclically, and fixes none of them, while y switches two elements and fixes the third one. Of course, the identity fixes all three elements. So the character is the one shown in the table.

Direct sum. A representation ρ of G on V is given. If V is the direct sum of invariant subspaces $W \oplus W'$, we can restrict ρ to these subspaces to obtain representations of G on W and W'. One says that ρ is the direct sum of these restrictions.

Suppose just one invariant subspace W is given. When we choose a basis of V beginning with a basis $(v_1, ..., v_k)$ of W, say $(v_1, ..., v_k; v_{k+1}, ..., v_n)$, the matrix R_g of rho_g will have the block form

$$\begin{pmatrix} A_g & B_g \\ 0 & D_g \end{pmatrix}$$

where A_g is a$k \times k$ matrix, The fact that the lower left block is zero is because, if w is in W, then $gw = \rho_g w$ is also in W.

I think of the block B_g as 'junk'. If that block is zero, the span W' of $(v_{k+1}, ..., v_n)$ will be invariant, and ρ will be a direct sum. But, even when W' exists, one needs to choose the basis carefully.

Maschke's Theorem asserts that every representation of G is a direct sum of irreducible representations. So when W is given, the invariant subspace W' always exists. In terms of matrices, there is a basis, as above, so that B_g is the zero matrix. This isn't obvious at all.

Projection formula. Let $\chi_1, ..., \chi_r$ be the irreducible characters of a group G, and let χ be another character. The irreducible characters form an orthonormal basis forthe space of all characters, and according to Maschke's Theorem, χ is a sum of irrducible characters. The projectcion formula computes this sum:

$$\chi = c_1\chi_1 + \cdots + c_r\chi_r$$

where $c_i = \langle \chi, \chi_i \rangle$.

Let's apply this formula when χ is the character χ_{perm}. Let $\chi_1 = \chi_{triv}$, $\chi_2 = \chi_{sign}$, $\chi_3 = \chi_{stand}$. Then

$$\langle \chi, \chi \rangle = \frac{1}{6}[(1) \cdot 3 \cdot 3 + (2) \cdot 0 \cdot 0 + (3) \cdot 1 \cdot 1] = 2$$

This tells us that χ is the sum of two irreducible representations. Next,

$$\langle \chi, \chi_1 \rangle = \frac{1}{6}[(1) \cdot 3 \cdot 1 + (2) \cdot 0 \cdot 1 + (3) \cdot 1 \cdot 1] = 1$$

and

$$\langle \chi, \chi_3 \rangle = \frac{1}{6}[(1) \cdot 3 \cdot 2 + (2) \cdot 0 \cdot (-1) + (3) \cdot 1 \cdot 0] = 1$$

Therefore $\chi = \chi_1 + \chi_2$.

As we saw above, the trivial representation corresponds to the invariant subspace of dimension 1 spanned by the vector $(1, ..., 1)^t$. What is the complementary invariant subspaceW' such that $V = W \oplus W'$? Here it is fairly easy to see that W' is the orthogonal space to W. This is true because the permutation matrices are orthogonal. For a general representation, when an invariant subspace is given, we need to have a positive definite hermitian form on V such that the operators ρ_g are unitary. We discuss this next time.