April 30, 2021

18.702 Comments on Problem Set 9

1. Chapter 15, Exercise 3.2. *(a polynomial that is irreducible over* $\mathbb{Q}[\sqrt[3]{2}]$*)*

The Eisenstein Criterion shows that the polynomial $f(x) = x^4 + 3x + 3$ is irreducible over $F = \mathbb{Q}$, as is the polynomial $x^3 - 2$. Then if β is a root of f and $\alpha = \sqrt[3]{2}$, β has degree 4 and α has degree 3, over F. Therefore $K = F[\beta, \alpha]$ has degree 12 over F, and degree 4 over $F[\alpha]$. This means that $f(x)$ remains irreducible over $F[\alpha]$.

2. Chapter 15, Exercise 3.7a. *(is* i *in the field* $\mathbb{Q}\big(\sqrt[4]{-2}\big)$*?)*

The answer is: No.

Let $F = \mathbb{Q}$, and let $\alpha = \sqrt[4]{-2}$ and $\beta = \sqrt[4]{2}$, so that $\alpha = \beta i$. If i were in the field $K = F[\alpha]$, then β would be in K too. The field $F[\beta]$ has degree 4 over F because $x^4 - 2$ is irreducible over F. Moreover, $F[\beta]$ is a subfield of $\mathbb{R}$, so it doesn't contain i. Therefore $F[\beta, i]$ has degree 8 over F, whereas its subfield $F[\beta i]$ has degree 4.

3. Chapter 15, Exercise 3.8. *(a condition for* α *and* β *to be algebraic)*

α and β are roots of the polynomial $x^2 - (\alpha + \beta)x + (\alpha\beta)$.

4. Chapter 15, Exercise 4.1. *(the irreducble polynomial for* $1 + \alpha^2$*)*

The elements $1, \alpha, \alpha^2$ form a basis for K over $F = \mathbb{Q}$, and $\alpha^3 = \alpha + 1$. To find the irreducible polynomial of $\beta = 1 + \alpha^2$, we compute its powers:

$\beta^2 = 1 + 2\alpha^2 + \alpha^4 = 1 + 2\alpha^2 + (\alpha^2 + \alpha) = 1 + \alpha + 3\alpha^2$,

$\beta^3 = \beta\beta^2 = (1 + \alpha^2)(1 + \alpha + 3\alpha^2) = (1 + \alpha + 3\alpha^2) + (\alpha + \alpha^2 + 3\alpha^3) = 1 + 5\alpha + 6\alpha^2$.

I got the equation $\beta^3 - 7\beta^2 - 5\beta + 10 = 0$. But it ery possibly that I made a mistake.