March 13, 2021

Comments on 18.702 Problem Set 3

1. *A finite group G operates on itself by conjugation. This operation produces a permutation representation of G.*
(a) *Determine the character χ_c of this representation.*

$\chi_c(g)$ is the number of group elements such that $hgh^{-1} = g$, i.e., such that $hg = gh$. It is the order of the centralizer of g: $\chi_c(g) = |Z(g)|$.

(b) *Let the conjugacy classes in G be $C_1, ..., C_k$, and let χ' be another character. Write $\langle \chi_c, \chi' \rangle$ as a sum over the conjugacy classes.*

Let g_i be an element of C_i. Let $c_i = |C_i| = |C(g_i)|$ and $z_i = |Z(g_i)|$. Then $c_i z_i = |G|$. So

$$< \chi_c \chi' >= \frac{1}{|G|} \sum_i c_i \chi_c(g_i) \chi'(g_i) = \frac{1}{|G|} \sum_i c_i z_i \chi'(g_i) = \sum_i \chi'(g_i)$$

(c) *Explain how to determine the decomposition of χ_c into irreducible characters by looking at the character table.*

Let $\chi_1, ..., \chi_k$ be the irreducible characters, as listed in the character table. Then $\chi_c = r_1\chi_1 + \cdots + r_k\chi_k$, where r_k is the sum of the entries in the row i of the character table.

For example, let G be the tetrahedral group T. Looking at the character table 10.4.14 in the text, one sees that $\chi_c = 4\chi_1 + \chi_2 + \chi_3 + 2\chi_4$.

As a check: The dimensions of the characters are $1, 1, 1$ and 3, respectively. So $4 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 3 = 12$, which is the order of the group.

2. Chapter 10, Exc. M.4 *(elements in the center)*

Suppose that z is in the center of G. Let ρ be a representation of G on V, let a be an eigenvalue of ρ_z, and let W be the subspace of V of vectors such that $\rho_z w = aw$. This is not the zero space, and it is an invariant subspace. To check this, we must show that for all g in G, and all w in W, $\rho_g w$ is in W, i.e., $\rho_z(\rho_g w) = a(\rho_g w)$. Since z is in the center and ρ is a homomorphism,

$$\rho_z(\rho_g w) = \rho_{zg} w = \rho_{gz} w = \rho_g \rho_z w = \rho_g a w = a \rho_g w$$

So if ρ is an irreducible representation, then $W = V$. This means that ρ_z is multiplication by a.

Conversely, let $\rho : G \longrightarrow GL(V)$ be a representation. If ρ_z is multiplication by a scalar, then it is in the center of $GL(V)$. The intersection of the kernels of the irreducible representations is the trivial subgroup $\{1\}$ of G. This is true because the kernel of the regular representation ρ^{reg} is $\{1\}$, and ρ^{reg} is the direct sum of irreducible representations. Therefore, if ρ_z is in the center of $GL(V)$ for every irreducible representation ρ, then z is in the cnter of G.

3. Chapter 11, Exc. 3.3 *(kernels of some homomorphisms)*

(b) Since every ideal of $\mathbb{R}[x]$ is principal, there is a single generator. It must be a polynomial in the kernel that has the lowest possible degree. The polynomial $f(x) = (x-(2+i))(x-(2-i)) = x^2-4x+5$ is in the kernel. Since $2+i$ isn't real, the kernel can't contain a linear or a constant polynomial. So f generates the kernel.

(e) The polynomials $u = y - x^2$ and $v = z - x^3$ generate the kernel. To see this, let $g(x,y,z)$ be in the kernel. Since u is monic in y, we may do division with remainder in $\mathbb{C}[x,y,z]$: $g = uq + r$, where r has degree zero in y, i.e., is independent of y, and is in the kernel. Next, we divide r by v in the ring $\mathbb{C}[x,z]$: $r = vq' + r'$, where r' is independent of y and of z. It is a polynomial in x in the kernel. Since x maps to t, the only such polynomial is the zero polynomial: $r' = 0$. Therefore $g = uq + vq'$.

4. Chapter 11, Exc. 3.9 *(unipotent and nilpotent elements)*

This is based on the power series expansion $(1+x)^{-1} = 1 + x + x^2 + \cdots$. If $x^n = 0$, then $(1+x)^{-1} = 1 + x + x^2 + \cdots + x^{n-1}$.

Let p be a prime. The binomial coefficients $\binom{p}{i}$ for $i = 1, ..., p-1$ are all divisible by p. Therefore, in characteristic p, $(1+a)^p = 1 + a^p$, $(1+a)^{p^2} = (1+a^p)^p = 1 + a^{p^2}$, etc.