18.702

Comments on Problem Set 2

1. Chapter 10, Exercise 6.5. (*the standard representation of* S_n)

As is true for any permutation representation, the trivial representation is the summand that corresponds to the invariant subspace U spanned by $(1, ..., 1)^t$. Since permutation matrices are orthogonal, and therefore unitary, $W = U^\perp$ is an invariant subspace. It is the space of vectors $(a_1, ..., a_n)^t$ such that $\sum a_i = 0$, and its dimension is $n - 1$.

Let w be an nonzero vector in W. In 18.701, in problem M1 of Chapter 4, I hope that you proved that the span of the orbit of w can have dimension $0, 1, n - 1$, or n, and that the first two cases occur when the vector is constant: $(a, a, ..., a)^t$. Since w is in W, the sum of its entries is zero. So, since w isn't zero, it isn't a constant vector. The dimension of its span is $n - 1$. Every nonzero vector w in W spans W. Therefore W is irreducible.

3. Chapter 10, Exercise M.9 *(Frobenius Reciprocity)*

As mentioned in the pset, this comes out when one writes what has to be proved carefully.

(a) To show that $ind\, S$ is a representation of G, you have to compute the four products $(ind\, S)_h(ind\, S)_g$, $(ind\, S)_g(ind\, S)_h$, $(ind\, S)_h(ind\, S)_{h'}$, and $(ind\, S)_g(ind\, S)_{g'}$.

The character $\chi_{ind\, S}$ of $ind\, S$ is zero on the coset aH, and $\chi_{ind\S}(h) = \chi_S(h)+$ $\chi_S(a^{-1}ha)$ when h is in H.

(c) The definition of $<,>$ shows that

$$< \chi_{ind\, S}, \chi_R >= \frac{1}{|G|}\left(\sum_h \chi_S(h)\chi_R(h) + \sum_h \chi_S(a^{-1}ha)\chi_R(h) \; + \; 0 \right)$$

Since $|G| = 2|H|$, the first sum, divided by $|G|$, is $\frac{1}{2} < \chi_S, \chi_{res\, R} >$. The only tricky part is the second sum. Since h and $a^{-1}ha$ are conjugate in G and since R is a representation of G, $\chi_R(h) = \chi_R(a^{-1}ha)$. So we can replace that sum by $\sum_h \chi_S(a^{-1}ha)\chi_R(a^{-1}ha)$, which is equal to $\sum_h \chi_S(h)\chi_R(h)$, summed in another order.

(d) Looking at the definition of $ind\, S$, one sees that $res(ind\, S) = S \oplus S'$. Frobenius Reciprocity tells us that

$$< ind\, S, ind\, S >=< S, res(ind\, S) >=< S, S \oplus S' >=< S, S > + < S, S' >$$

The right side of this equation is 1 if $S \not\approx S'$ and 2 if $S \approx S'$.

4. *Determine the character table of the symmetric group S_5.*

Two elements are conjugate in the symmetric group if their cycle decompositions have the same sizes. The cycle decompositions are indicated by dots in the table.

The character table consists of the first seven rows below. The classes to the left of the second vertical line are the even conjugacy classes. Because there are so many cross-checks, there is more than one way to determine the table.

size	(1)	(15)	(20)	(24)	(10)	(20)	(30)
	(.)	(..)(..)	(...)	(.....)	(..)	(...)(..)	(....)
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	4	0	1	-1	2	-1	0
χ_4	4	0	1	-1	-2	1	0
χ_5	5	1	-1	0	1	1	-1
χ_6	5	1	-1	0	-1	-1	1
χ_7	6	-2	0	1	0	0	0
perm	5	1	2	0	3	0	1
(..)	10	2	1	0	4	1	0

Notice that the character values are integers. It isn't hard to verify this fact. A permutation is conjugate to its inverse. For instance, when g is a 4-cycle, i and $-i$ appear as eigenvalues of ρ_g the same number of times.

In the table, χ_2 is the sign character, χ_3 is determined by decomposing the permutation representation that corresponds to the operation of S_5 on the set of five indices. That character is labelled *perm*. Then $\chi_4 = \chi_2\chi_3$ (as discussed in Problem 2).

Next, we compute the permutation representation for the operation by conjugation on on the conjugacy class (..). Its character is the last row of the table. To compute the value of this character on the 3-cycle $(1\,2\,3)$, for example, one has to determine the number of transpositions $(i\,j)$ that commute with a 3-cycle: $(1\,2\,3)(i\,j) = (i\,j)(1\,2\,3)$. The answer is that the transposition $(4\,5)$ is the only one. So the value of the character in that position is 1.

Using the projection formula, one sees that the character labelled (..) is the sum of three irreducible characters, including χ_1 and χ_3. So $(..) - \chi_1 - \chi_3$ is an irreducible character. It is labelled χ_5, and $\chi_6 = \chi_5\chi_2$. Then χ_7 is determined by orthogonality.